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CONFORMALLY FLAT SEIFERT MANIFOLDS

I.O.Belegradek

Abstract

We construct uniformizable conformally flat structures on certain $SL_2(\mathbb{R})$ Seifert manifolds. It has been known that some of the $SL_2(\mathbb{R})$ Seifert manifolds admit uniformizable conformally flat structures ([1], [2], [3], [4]). On the other hand there are $SL_2(\mathbb{R})$ Seifert manifolds that do not admit any uniformizable conformally flat structure [5]. The present paper partly fills the gap between these results.

Key words and phrases: conformally flat structure, Seifert manifold, Kleinian group, hyperbolic structure.

§1. INTRODUCTION

The first examples of conformally flat structures on $SL_2(\mathbb{R})$ Seifert manifolds (without singular fibers) were constructed by M.Gromov, H.B.Lawson, W.Thurston [1], M.Kapovich [2] and N.H.Kuiper [3]. They considered Seifert manifolds which were the total spaces of the circle bundles over closed orientable surfaces. It is shown in [3] that if the genus g and Euler number e of such a manifold satisfy the conditions $g \geq 3$, $|e| \leq (2g-2)/3$, then it admits a conformally flat structure.

Recently Feng Luo [4] has constructed examples of conformally flat Seifert manifolds with two singular fibers over an orientable base of arbitrary genus. The Euler numbers of that manifolds satisfy the inequality $|e| < 1/2$.

The present paper contains the construction of conformally flat structures on certain Seifert manifolds with an arbitrary number of singular fibers. Feng Luo and the author have obtained their results independently. The classes of manifolds considered by Feng Luo and by the author are different but overlap.

The main result of the paper is as follows.

Theorem. *Let M be a closed orientable Seifert manifold of genus g with Euler number e. Let $r = 2g$ if the base of M is orientable, and $r = g$ otherwise. Suppose that one of the following holds:*

(1) *M has at least one singular fiber, $r \geq 6$, $|e| \leq (r-1)/8$,*

(2) *M has no singular fibers, $r \geq 8$* and $|e| \leq [r/8]$.

Then M admits a uniformizable conformally flat structure.

Note that the case "the base M is orientable and (2) holds" is covered by [3]. The theorem gives a partial answer to the following

Question. [4] Given a hyperbolic 2-orbifold X, is there an $\varepsilon > 0$ so that all Seifert manifolds over X with Euler number e, $|e| < \varepsilon$, admit conformally flat structures?

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The Möbius group G uniformizing M in the theorem acts freely and discontinuously in a certain domain $\Omega \subseteq S^3$; the limit set of G is $S^3 - \Omega$ and forms a circle in S^3. The group G is isomorphic to $\pi_1(X)$, where X is the base orbifold of M. Therefore, if X has singular points, G has elements of finite order (i.e., elliptic elements); moreover, there is a natural one-to-one correspondence between the singular points of X and the conjugacy classes of elliptic elements in G. Note that in G every elliptic element has no fixed point in S^3 and has the only fixed point in H^4 under the natural action of G as a discrete group of isometries.

The hyperbolic orbifold H^4/G is isomorphic to a plane bundle over a 2-orbifold X and contains M as its unit circle bundle over X. If X has exactly m singular points, the singular locus of H^4/G consists of m points; some neighborhood of any singular point is a cone over a lens space, so the underlying space of H^4/G is not a manifold.

If X has no singular point, H^4/G is a hyperbolic manifold. The first examples of such manifolds were given in [1], [2] and [3]. A notable feature of our method is that it can also produce hyperbolic manifolds which are plane bundles over nonorientable surfaces.

Recently M.Kapovich [5] has found a necessary condition for the existence of a complete hyperbolic structure on a plane bundle over a closed orientable surface. As a by-product he has obtained a necessary condition for existence of a uniformizable conformally flat structure on a $SL_2(\mathbf{R})$ Seifert manifold. It has been unknown if there exist a $SL_2(\mathbf{R})$ Seifert manifold with a nonuniformizable conformally flat structure (i.e., such that the development map is surjective).

Finally, I would like to express my gratitude to my adviser N.A.Gusevskii for fruitful discussions, his comments and information provided.

§2. PRELIMINARIES

2.1 *The Möbius geometry.* Let $M(n)$ be the group of all orientation-preserving Möbius (i.e., conformal) transformations of $S^n = \mathbf{R}^n \cup \{\infty\}$, $n > 2$. We say that a Möbius group G acts discontinuously at $x \in S^n$ if there is a neighborhood V of x such that $g(V) \cap V = \varnothing$ for all but finitely many $g \in G$. The set of points at which G acts discontinuously is called the set of discontinuity of G and denoted by $\Omega = \Omega(G)$. The complement of Ω in S^3 is said to be the limit set of G and denoted by $\Lambda = \Lambda(G)$.

A fundamental domain for G is an open set $F \subseteq \Omega(G)$ such that

(i) $g(F) \cap F = \varnothing$, for $g \in G - \{\mathrm{id}\}$;

(ii) $\bigcup_{g \in G} g(\bar{F}) = \Omega(G)$.

A Möbius (or conformally flat) structure on an n-dimensional manifold is a maximal atlas such that the transition functions are the restrictions of Möbius mappings.

A traditional way of constructing a Möbius structure on a manifold is uniformization: suppose a Möbius group G acts freely and discontinuously in a domain $D \subseteq S^n$; then a Möbius structure naturally arises on the quotient manifold D/G.

We consider S^n as the boundary of the open ball B^{n+1} with the hyperbolic metric. Every Möbius transformation can be naturally extended to an isometry of B^{n+1} and the

group $M(n)$ can be identified with the group $\mathrm{Isom}(B^{n+1})$ of all orientation-preserving hyperbolic isometries.

We need the following classification of nonidentitical Möbius transformations. Every Möbius transformation g has at least one fixed point in the closure of B^{n+1}. If g has a fixed point in B^{n+1} then g is elliptic; if g is not elliptic and has exactly one fixed point in S^n, then g is parabolic; otherwise, g is loxodromic and has exactly two fixed points in S^n. A loxodromic transformation which is conjugate to a dilatation is called hyperbolic.

Denote by $\mathrm{Fix}(g)$ the set of fixed points of g in S^n. If $n = 3$ and g is elliptic, $\mathrm{Fix}(g)$ is a round circle or empty.

We use the word "round" in the following sense. A round ball (sphere) is the image of the unit ball (sphere) under a Möbius transformation .

A round annulus is a topological annulus in S^3 lying on a round 2-sphere and bounded by two round circles. Any round annulus A is Möbius equivalent to a plane concentric annulus with radii r and R; it is known that $\log(R/r)$ depends on A only; this number is the conformal module of A. Any two round annuli are Möbius equivalent iff their moduli are equal.

2.2 *Seifert manifolds.* By a Seifert manifold we mean a closed orientable manifold which is the total space of a circle bundle over a 2-orbifold whose singular points are conic (cf.[6, §13]). Such a bundle is called a Seifert bundle. It is orientable iff the base orbifold is orientable. Define the signature of a Seifert bundle as the tuple $(\varepsilon, r, b, (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m))$, where the meaning of the parameters is as follows. The parameter ε tells us about the orientability of the base orbifold: $\varepsilon = \mathfrak{O}$ if it is orientable, and $\varepsilon = \mathfrak{N}$ otherwise. Let g be the genus of the base orbifold; then $r = 2g$ if the base orbifold is orientable, and $r = g$ otherwise. The definitions of b, α_i, β_i can be found in [7, §6]. The integer b reflects the "degree of twisting" of a regular fiber. The natural number m is the number of singular fibers; the integers α_i, β_i are the Seifert invariants of ith singular fiber, α_i and β_i are coprime, $0 < \beta_i < \alpha_i$. The signature determines a Seifert bundle up to bundle equivalence (see [8, §5]).

The Euler number e of the bundle is equal to $-b - \sum_{i=1}^{m} \beta_i/\alpha_i$. The Euler characteristic χ of the base orbifold is equal to $2 - r - m - \sum_{i=1}^{m} 1/\alpha_i$.

The Euler numbers of Seifert bundles have the naturality property. In particular, if $\tilde{M} \to M$ is a covering of Seifert bundles induced by an n-sheeted covering $\tilde{X} \to X$ of its base 2-orbifolds, $e(\tilde{M}) = n\,e(M)$.

Since also $\chi(\tilde{X}) = n\chi(X)$, we have $e(\tilde{M})/\chi(\tilde{X}) = e(M)/\chi(X)$ (see [9]). A change of the orientation of a Seifert manifold M gives the signature

$$(\varepsilon, r, -b-m, (\alpha_1, \alpha_1-\beta_1), \ldots, (\alpha_m, \alpha_m-\beta_m));$$

therefore, $e(-M) = e(M)$.

Any Seifert manifold admits a geometric structure (in the sense of W.Thurston). The geometry is determined by the numbers e and χ according to the following table:

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbf{R}$	$\mathbf{E}^3$	$H^2 \times \mathbf{R}$
$e \neq 0$	S^3	Nil	$SL_2(\mathbf{R})$

If $\chi < 0$ (i.e., the base orbifold is hyperbolic), $\pi_1(M)$ determines M up to bundle equivalence. If $\chi \geq 0$, $\pi_1(M)$ is abelian-by-finite.

A classification of Seifert manifolds [10] is based on cutting them into "simple" pieces. An exposition of this method can be found in [11, p.116]. In this book a standard presentation of $\pi_1(M)$ in terms of the parameters of the signature of M is given.

Since the group of homeomorphisms of S^1 retracts onto $O(2)$, every Seifert manifold with the base X is the unit circle bundle of a 2-plane bundle over X.

§3. GROUPS OF NECKLACES

3.1. In this section we prepare tools needed for the principal construction. These preparatory results stem from to the constructions developed in [1, 2, 3].

Consider a finite number of circles of the form $\partial D^2 \times \{x\}$ on a closed solid torus $D^2 \times S^1$, where $x \in S^1$; we call them edges. The edges divide the torus $\partial D^2 \times S^1$ into annuli which are called sides. We call a homeomorphic image (with the edges and sides marked) in S^3 of the marked solid torus a *necklace* if the images of the sides are round annuli. The images of the edges (the sides) of the marked solid torus are called the *edges* (the *sides*) of the necklace. The complement of a necklace in S^3 is called the *conecklace* of it.

Let σ be a permutation of the set of sides of a necklace of order two without fixed points. Suppose for every side s an orientation-preserving Möbius transformation g_s is chosen in such a way that $g_{\sigma(s)} = g_s^{-1}$ and g_s maps s onto $\sigma(s)$. We say that g_s pairs the sides s and $\sigma(s)$. The Möbius group generated by all g_s's is said to be the *group of the necklace (corresponding to the generators)*. A necklace together with such a group is called an *equipped* necklace.

We say that edges e and e' are paired if e' is $g_{s_1}(e)$ or $g_{s_2}(e)$, for the sides s_1 and s_2 whose common edge is e. It is easy to see that for any edge e one and only one of the following possibilities holds: e is paired only with itself; e is paired with some other edge which is paired only with e; e is paired with exactly two edges. We write $e \sim e'$ if there is a sequence of edges $e_0, \ldots, e_k$ such that $e_0 = e$, $e_k = e'$ and e_i is paired with e_{i+1}, for $i < k$. Clearly, $\sim$ is an equivalence relation on the set of all edges; every equivalence class can be cyclically ordered in such a way that neighbors are paired and nonneighbors are not. The equivalence classes are called *cycles of edges*. Let $(e_0, \ldots, e_n)$ be a cycle of edges, $e_{i+1} = g_i(e_i)$. The transformation $g = g_n \circ \ldots \circ g_0$ is called a *cyclic transformation* of the edge e_0; clearly $g(e_0) = e_0$.

Under certain conditions one can construct a new necklace from the two given. Suppose a side s_i of a necklace N_i coincides with the intersection of a round sphere containing s_i with the closure of the conecklace F_i of the necklace N_i, $i = 1, 2$. If the round annuli s_1 and s_2 are Möbius equivalent, there exist a Möbius transformation φ mapping s_1 onto s_2 such that $F_1 \cup_\varphi F_2$ is the conecklace of some necklace N. We say that N is *copasted* from N_1 and N_2 along the sides s_1 and s_2. If N_1 and N_2 are equipped necklaces, an induced equipment on N arises.

An equipped necklace is said to be admissible if

(1) for every side, the conecklace and its image under g_s are disjoint;

(2) for every cycle of edges there exist coprime integers α, β with $0 \leq \beta < \alpha$ such that

(i) the sum of angles at the edges of the cycle is equal to $2\pi/\alpha$;

(ii) any cyclic transformation of any edge in the cycle is conjugate to a Möbius transformation which acts on the edge as a rotation of $2\pi\beta/\alpha$ by a Möbius transformation preserving the edge.

The following is the main result of this section.

Theorem 3.2. *Let N be an admissible necklace with group G and conecklace F. Then*

(1) G acts freely and discontinuously in $\Omega = \bigcup_{g \in G} g(\overline{F})$; the conecklace F is a fundamental domain for G the manifold Ω/G is closed, orientable and admits a uniformizable conformally flat structure.

(2) If N is unknotted, then Ω/G is an $H^2 \times \mathbf{R}$ or $SL_2(\mathbf{R})$ Seifert manifold.

(3) If N is knotted, then Ω/G is an $H^2 \times \mathbf{R}$ or $SL_2(\mathbf{R})$ Seifert manifold modified by removing a tubular neighborhood of a regular fiber and replacing it with F.

Proof. (A) We begin with a proof of (1).

The group G acts in H^4 as a group of isometries. Let P be the convex hull of F in H^4. By the Poincaré-Maskit Theorem [12, p.73], G is a discrete group with fundamental polyhedron P. Therefore, G acts discontinuously in Ω, and F is a fundamental domain for G. To prove (1) it suffices to show that G acts freely in Ω.

The condition (2) of the definition of an admissible necklace guarantees that the cyclic transformations of the polyhedron P act freely in S^3. Show that any elliptic element in G is conjugate to a power of a cyclic transformation of P. Let h be a conjugate of the elliptic element such that h has a fixed point x in $\overline{F}$; clearly $x \in \partial P$. In fact, x lies on an edge of P. (Indeed, if x is an interior point of a side s of P then $h(P)$ meets $g_s^{-1}(P)$; hence, $h = g_s^{-1}$. Therefore $g_s^{-1}(x) = x$, a contradiction.) It is easy to see that different edges of P are disjoint, so the edge e containing x is unique. For any $f \in G$, if $f(\overline{F})$ meets e then $f(e)$ is disjoint from e or f is a power of a cyclic transformation of e. Since $h(x) = x \in e$, h is a power of a cyclic transformation of e.

(B) Now we prove (2). To show that $M = \Omega/G$ is a Seifert manifold it suffices to prove that $\pi_1(M)$ contains an infinite normal cyclic subgroup [13]. Since the covering $\Omega \to M$ is regular, the group $\pi_1(\Omega)$ is a normal subgroup of $\pi_1(M)$. We show that $\pi_1(\Omega) \cong \mathbf{Z}$; moreover, Ω is a solid torus. Let F_1 be the closure of F and F_{m+1} the union of all the sets of the form $g(F_m)$, where $g \in G$. $F_m \cap g(F_m) = \varnothing$. Let

$$D_m = \{x \in \mathbf{R}^2 : |x| \leq m/m+1\},\ B_m = D_m \times S^1.$$

Since every F_m is a solid torus and $F_{m+1} - F_m$ is homeomorphic to $T^2 \times (0, 1]$, every homeomorphism from F_m onto B_m can be extended to a homeomorphism from F_{m+1} onto B_{m+1}. Therefore, there is a homeomorphism from $\bigcup_{m=1}^{\infty} F_m = \Omega$ onto the solid torus $\bigcup_{m=1}^{\infty} B_m$.

To complete the proof of (2), it suffices to show that the base orbifold of M is hyperbolic. Suppose the contrary. Then M is an S^3, $S^2 \times \mathbf{R}$, Nil, or $\mathbf{E}^3$ Seifert manifold, hence $\pi_1(M)$ is abelian-by-finite. Therefore, $G \cong \pi_1(M)/\pi_1(\Omega)$ is also abelian-by-finite. As is well known, a discrete Möbius group is abelian-by-finite iff it is elementary. So $\pi_1(\Omega)$ is trivial, a contradiction.

In fact, the group G completely determines the base orbifold X, as the following result of some independent interest shows.

Proposition 3.3. *Let a group G uniformize a conformally flat structure on a Seifert manifold M with a hyperbolic base orbifold X. Suppose one of the following holds:*

(a) *M is an $SL_2(\mathbf{R})$ Seifert manifold;*

(b) *M is an $H^2 \times \mathbf{R}$ Seifert manifold, and G is the group of an admissible necklace.*

Then $\pi_1(X) \cong G$.

Proof. Since G is the quotient group of $\pi_1(M)$ modulo the normal cyclic subgroup $\pi_1(\Omega)$ and every normal cyclic subgroup of $\pi_1(M)$ is generated by t^n, for t corresponding to a regular fiber, the presentation of G can be obtained from that of $\pi_1(M)$ by adding the relation $t^n = 1$. It suffices to show that $t = 1$ in G. To prove this, suppose the contrary; then t is an elliptic element, so $\mathrm{Fix}(t)$ is either the empty set or a round circle.

Let $\mathrm{Fix}(t) = \varnothing$. Considering a two-sheeted covering of M, one can assume that X is orientable; then t generates the centre in $\pi_1(M)$ and G. For every ε it is easy to construct a loxodromic element $g_\varepsilon \in G$ with the distance between the fixed points of g_ε less than ε. Since t acts in S^3 freely and discontinuously, $t(\mathrm{Fix}(g_\varepsilon))$ and $\mathrm{Fix}(g_\varepsilon)$ are disjoint, for some ε; so $t g_\varepsilon \neq g_\varepsilon t$, a contradiction.

Let $\mathrm{Fix}(t)$ be a round circle. Since the limit set of G is a circle in S^3 [2] and t acts freely in Ω, $\mathrm{Fix}(t)$ is a limit set of G. Therefore Ω is homeomorphic to $H^2 \times S^1$ and M is an $H^2 \times \mathbf{R}$ Seifert manifold. Since the geometry on M is unique, (a) is impossible and (b) holds. Then, as we showed in (A), any elliptic element in G acts freely in S^3, a contradiction.

The proof of Proposition 3.3 is complete.

R e m a r k 3.4. In Proposition 3.3 the second part of condition (b) is essential: for any closed orientable hyperbolic surface X, it is easy to construct a group uniformizing the $H^2 \times \mathbf{R}$ Seifert manifold $X \times S^1$ and isomorphic to $\pi_1(X) \times \mathbf{Z}_n$.

3.5. Now we give a more direct proof of (2) which gives the invariants of M in terms of the geometry of N. The idea of this proof was suggested by N.A.Gusevskii.

Let g_e be a cyclic transformation of an edge e of the necklace N. Construct a g_e-invariant tubular neighborhood of e. By (2i) of the definition of an admissible necklace, the sum of angles at the edges of the cycle containing e is equal to $2\pi/\alpha$; so g_e rotates the sides abutting e by $2\pi/\alpha$. Taking into account (2ii), we have $g_e = h_e \circ \varphi_e \circ \psi_e \circ h_e^{-1}$, where φ_e is a rotation about e by $2\pi/\alpha$, ψ_e is a rotation about the axis of rotation of the circle e by $2\pi\beta/\alpha$, and h_e preserves e. Let ε be less than the radius of e; choose a point $x \notin e$ in the ε-neighborhood of e. Consider the circle containing x whose axis of rotation is e and the torus that is the result of the rotation of this circle about the axis

of rotation of e. This torus T_e bounds $\varphi_e \circ \psi_e$-invariant tubular neighborhood W_e of e. Then $U_e = h(W_e)$ is a g_e-invariant tubular neighborhood of e.

Let $(e_0, \ldots, e_n)$ be the cycle of edges, $e_0 = e$, $g_i(e_i) = e_{i+1}$. Then $g_{e_k} = g_{k-1} \circ \ldots \circ g_0 \circ g_n \circ \ldots \circ g_k$ is a cyclic transformation of e_k, and $U_{e_k} = (g_k \circ \ldots \circ g_0)(U_e)$ is a g_{e_k}-invariant tubular neighborhood of e_k. So for every edge e we have a g_e-invariant tubular neighborhood, and all the neighborhoods of the edges in every cycle are G-equivalent. Choosing the ε's small enough, we can consider neighborhoods with disjoint closures.

The images of the wedges $U_e \cap \overline{F}$ under the transformations $g_i \circ \ldots \circ g_0 \circ g_e^m$, $i \leq n$, $m < \alpha$, fill out the union of all U_e's.

Denote by p the covering $\Omega \to M = \Omega/G$. Let E be the set of all edges of N.

Clearly, the manifold M can be obtained by pasting to $M_0 = p(\overline{F} - \bigcup_{e \in E} U_e)$ the solid tori $p(\overline{U}_e)$ along the tori $p(\partial U_e)$.

(A) First we study the manifold M_0. Consider a homeomorphism of the solid torus $\overline{F} - \bigcup_{e \in E} U_e$ onto $D^2 \times S^1$ such that the images of the circles $\partial U_e \cap \partial F$ are of the form $\{x\} \times S^1$, where $x \in \partial D^2$. So the torus $\partial D^2 \times S^1$ is divided into annuli in such a way that the images of the annuli $\partial U_e \cap \overline{F}$ alternate with the images of the annuli containing in $\partial F - \bigcup_{e \in E} U_e$. The pairing transformations of N induce pairing transformations for the set of annuli of the latter type on the torus $\partial D^2 \times S^1$; we shall use the same notation for the induced pairing transformations as for the pairing transformations of N, that is s, $\sigma(s)$, g_s.

For every annulus s, define a homeomorphism f_s mapping s onto $\sigma(s)$ with $f_{\sigma(s)} = f_s^{-1}$. Let $s = p_s \times S^1$, where p_s is an arc of ∂D^2. Fix orientations on ∂D^2 and S^1. Let ρ_s and τ_s be homeomorphisms of p_s onto $p_{\sigma(s)}$ and suppose that ρ_s preserves and τ_s changes the orientation. Let $r: S^1 \to S^1$ be a reflection. Put $f_s = \rho_s \times r$ if g_s changes the orientation of the circle $\{\mathit{end}\, p_s\} \times S^1$, and $f_s = \tau_s \times \mathrm{id}_{S^1}$ otherwise.

It is well known [10, p.384] that the quotient manifold of $D^2 \times S^1$ modulo the identifications f_s is an orientable manifold Σ which is a circle bundle over a compact surface V with boundary. If all the f_s's are $f_s = \tau_s \times \mathrm{id}_{S^1}$, the surface V is orientable and $\Sigma = V \times S^1$; otherwise V is nonorientable and the bundle $S^1 \to \Sigma \to V$ is nonorientable.

We prove that the quotient manifolds of $D^2 \times S^1$ modulo f_s and g_s are homeomorphic. It suffices to construct homeomorphism Φ of $D^2 \times S^1$ onto itself transforming f_s to g_s. First define Φ on $\partial D^2 \times S^1$ in such a way that $\Phi|_s = \mathrm{id}_s$, $\Phi|_{\sigma(s)} = g_s \circ f_s^{-1}$ and the automorphism of $\pi_1(\partial D^2 \times S^1)$ induced by Φ is the identity mapping. Then Φ is isotopic to the identity homeomorphism of $\partial D^2 \times S^1$; hence, Φ can be extended to $D^2 \times S^1$.

(B) Now we investigate how $p(U_e)$ is pasted to M_0. Let C be a cycle of edges. Denote by A_e the image of the annulus $\partial U_e \cap \overline{F}$ on $\partial D^2 \times S^1$, $e \in C$. Consider the circle $\partial D^2 \times \{\varphi\}$, where $\varphi \in S^1$ is a fixed point of the reflection r. Denote $\overline{d} = \bigcup_{e \in C} A_e \cap (\partial D^2 \times \{\varphi\})$. Let $\overline{t}$ be a boundary circle of one of the A_e 's. Denote by d', t' the images and of $\overline{d}$ and $\overline{t}$ in Σ. The loops d' and t' form a base of the fundamental group of the boundary torus of Σ; t' is a fiber of the bundle $S^1 \to \Sigma \to V$.

Let O_e be a point on the torus ∂U_e corresponding to the point on $\partial D^2 \times S^1$ at which $\overline{d}$ meets $\overline{t}$. Let $\tilde{d}$ and $\tilde{t}$ be the h_e-preimages of the liftings of d' and t' on the torus ∂U_e beginning at O_e; $\tilde{d}$ and $\tilde{t}$ are loops on T_e.

Choose coordinates (λ, θ) on the torus T_e, λ, $\theta \in [0, 2\pi)$, in such a way that $h_e^{-1}(O_e)$ is the origin, the invariant circles for φ_e and ψ_e are the coordinate axes λ and θ, and the directions of the axes coincide with those of the rotations. Note that the subsets of T_e defined by inequalities $0 < \lambda < 2\pi/\alpha$ and $0 < \theta < 2\pi/\alpha$ are fundamental domains for the group Γ generated by $\varphi_e \circ \psi_e$.

Clearly, $\tilde{t}$ is defined by the equation $\lambda = 0$, and $\tilde{d}$ is homotopic to the curve $\{(\lambda, \theta) : \theta = (\alpha b + \beta)\lambda,\ 0 \le \lambda \le 2\pi/\alpha\}$ for some integer b.

Let $\tilde{y}$ be the curve $\theta = 0$, and let $\tilde{x}$ be the curve

$$\{(\lambda, \theta) : \lambda = q\theta,\ 0 \le \theta \le 2\pi/\alpha\},$$

where q is the only integer with $q\beta \equiv 1 \pmod{\alpha}$, $0 < q < \alpha$.

Denote the solid torus W_e/Γ by W^e, $T^e = \partial W^e$. Let y, x, d, t, be the images of $\tilde{y}$, $\tilde{x}$, $\tilde{d}$, $\tilde{t}$ under the covering $T_e \to T^e$. The loops y and x form a base of $\pi_1(T^e)$, and y is null-homotopic in W^e. Analogously, d and t form a base of $\pi_1(T^e)$. It easy to see that in this group $y = d^{\alpha} t^{-\beta - \alpha b}$, $x = d^{q} t^{-k - q b}$, where $q\beta = 1 + k\alpha$.

A summary of (A) and (B). One can imagine the pasting of $p(\overline{U}_e)$ to M_0 along $p(\partial U_e)$ as such a pasting of W^e to Σ that the loops d, t are identified with the loops d', t'. So the pasting of $P(U_e)$ to M_0 is equivalent to the standard construction of a Seifert manifold [11, p.116]. A presentation of $\pi_1(M)$ can be obtained from that of $\pi_1(M_0)$ by the adding of the defining relations $d^{\alpha_c} = t^{\beta_c + \alpha_c b_c}$, where the integers α_c, β_c, b_c correspond to a cycle of edges c. Thus the Seifert invariants of M are the integers α_c, β_c (up to sign), and the invariant b for M is a sum, the summands of which are b_c (up to sign); c runs over the set of all cycles. The invariants ε and r for M are the same as the ones for the surface V.

So we have a constructive proof of (2).

3.6 Now it is easy to prove (3). Consider the necklace N and its tubular neighborhood homeomorphic to a solid torus. Cutting F by the boundary torus of this neighborhood, we get two components. The first one is homeomorphic to F. As in the constructive proof of (2) above, one can show that the quotient of the second component by G is an $H^2 \times \mathbf{R}$ or $SL_2(\mathbf{R})$ Seifert manifold without a tubular neighborhood of a regular fiber.

The proof of Theorem 3.2 is complete.

3.7. The group G of an admissible necklace N acts in H^4 as a discrete group of isometries. The convex hull of the conecklace F is a fundamental polyhedron P of the group G in H^4. The singular locus of the hyperbolic orbifold H^4/G is the finite set of the points corresponding to the fixed points of cyclic transformations of the necklace. In particular, if every cyclic transformation is the identity mapping, H^4/G is a hyperbolic manifold.

If N is unknotted, then the orbifold H^4/G is isomorphic to a plane bundle over a 2-orbifold X and contains M as the unit circle bundle over X. Indeed, P admits a natural G-invariant disk fibration whose restriction on S^3 is a G-invariant circle fibration of F. This disk fibration induces a structure of a disk bundle on $(H^4 \cup \Omega)/G$ whose restriction on the boundary is the circle bundle M over X.

Since the orbifold H^4/G is orientable, this plane bundle is orientable whenever so is X.

If N is knotted, the orbifold H^4/G is isomorphic to a plane bundle over a 2-orbifold X which has been modified as follows: at the regular point of X, we remove a local trivialization $D^2 \times \mathbf{R}^2$ from the bundle and paste in a copy of the set P along the boundary $S^1 \times \mathbf{R}^2$. Indeed, a neighborhood U of ∂P in P admits a G-invariant disk fibration, whose restriction on S^3 is a G-invariant circle fibration of a neighborhood of ∂F in F. This disk fibration induces a structure of a disk bundle on U/G. The induced bundle is a disk bundle over a 2-orbifold without a neighborhood of a regular point.

Show that if the orbifold H^4/G has singular points, its underlying space Y is not a manifold. Any elliptic element g in G generates the isotropy group I_x of the fixed point x of g. Let B_x be a round ball containing x and precisely invariant under I_x in G. Then B_x/I_x is a neighborhood of a singular point of H^4/G. Since B_x/I_x is a cone over a lens space, Y is not a manifold.

§4. EXAMPLES OF ADMISSIBLE NECKLACES

Example 4.1. (N.A.Gusevskii). Every $H^2 \times \mathbf{R}$ Seifert manifold has conformally flat structure because the product metric is conformally Euclidean. For such a manifold we construct an admissible necklace whose group uniformizes this manifold. The construction is analogous to that of a fundamental polygon of a Fuchsian group of prescribed signature (see [14, p.235]). Suppose the signature of an $H^2 \times \mathbf{R}$ Seifert manifold is $(\varepsilon, r, b, (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m))$. First we construct an equipped necklace which will be used several times below for constructing other necklaces. Let B a unit 2-disk with the hyperbolic metric. In B draw a regular Euclidean polygon of $2p + m$ sides inscribed in a circle centered at the origin O and of radius R, where $0 < R < 1$ and p is a nonnegative integer, $2p + m \ge 3$, p is even if $\varepsilon = 0$. Now replace each side by the hyperbolic arc connecting its endpoints. Let $H_i H_{i+1}$ be the curved side labelled $2p + i$ $(i = 1, \ldots, m)$ and M_i its midpoint. The interior angle subtended by $H_i H_{i+1}$ at M_i is π. If K_i is the point of intersection of the ray OM_i and the unit circle, the angle $H_i K_i H_{i+1}$ is 0. Therefore, there is a point N_i on the ray $M_i K_i$ such that the interior angle $H_i N_i H_{i+1}$ is $2\pi/\alpha_i$. We construct the new sides $H_i N_i$ and $N_i H_{i+1}$, $i = 1, \dots, m$, and now have a $(2p + 2m)$-gon P_R. Let $\tau(R)$ be the sum of the angles of P_R other than $H_i N_i H_{i-1}$, $i = 1, \dots, m$. It is clear that $\tau(R)$ varies continuously with R and $\tau(R)$ tends to 0 as R tends to 1. The polygon P_R approaches a Euclidian $(2p + 2m)$-gon, as R tends to 0; hence,

$$\tau(R) \to (2p + 2m - 2)\pi - 2\pi \sum_{i=1}^{m} 1/\alpha_i .$$

(If $p = r$, the lastexpression is equal to $2\pi(1 - \chi(X))$. Since $\chi(X) < 0$, this gives $\tau(0) > 2\pi$. It follows that in the case $p = r$ there is a ρ, $0 < \rho < 1$, for which $\tau(\rho) = 2\pi$.)

We shall consider B as a plane disk in R^3. Revolving $B - P_R$ about ∂B, we get a necklace N_R with $2p + 2m$ sides. Define side pairing transformations for N_R as follows. In the case $\varepsilon = \mathfrak{D}$ the integer p is even; put $\sigma = \sigma_1 \sigma_2$, where

$$\sigma_1 = (13)(24) \dots (2p-3, 2p-1)(2p-2, 2p),$$

$$\sigma_2 = (2p+1, 2p+2) \dots (2p+2m-1, 2p+2m).$$

In the case $\varepsilon = \mathfrak{N}$ put $\sigma = (12) \dots (2p + 2m - 1, 2p + 2m)$.

For $j = 1, \dots, 2p + 2m$ let I_j be the inversion in the sphere containing a side j; R_j, the reflection mapping a side j onto the side $\sigma(j)$; Ψ_j, the rotation by π about the straight line incident to O and to the center of the sphere containing the side $\sigma(j)$. Denote by Φ_i the rotation about ∂B by $2\pi\beta_i/\alpha_i$ in the positive direction.

We denote the transformation that pairs the sides j and $\sigma(j)$ by g_j. For $i = 1, \dots, m$, put

$$g_{2p+2i-1} = \Phi_i \circ R_{2p+2i-1} \circ I_{2p+2i-1}, \; g_{2p+2i} = g_{2p+2i-1}^{-1}.$$

If $\varepsilon = \mathfrak{N}$, for $j = 1, 2, 5, 6, \dots, 2r-3, 2r-2$, put

$$g_j = R_j \circ I_j, \; g_{j+2} = g_j^{-1}.$$

If $\varepsilon = \mathfrak{N}$, for $j = 2, 4, \dots, 2r$, put

$$g_j = \Psi_j \circ R_j \circ I_j, \; g_{j-1} = g_j^{-1}.$$

So we have constructed an equipped necklace N_R; we say that N_R has signature $(\varepsilon, p, (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m))$.

The necklace has m cycles of edges of length 1 and one cycle of length $2p + m$. The cyclic transformation of the edge of the ith "short" cycle rotates this edge by $2\pi\beta_i/\alpha_i$, $i = 1, \dots, m$. The cyclic transformation of an edge of the "long" cycle rotates the edge by $2\pi \sum_{i=1}^{m} \beta_i/\alpha_i$. Since $-b - \sum_{i=1}^{m} \beta_i/\alpha_i = e = 0$, the latter rotation is the identity. The angle at the edge of the ith "short" cycle is $2\pi/\alpha_i$. If $R = \rho$ and $p = r$ the sum of the angles at the edges of the "long" cycle is 2π; hence the necklace N_ρ with signature $(\varepsilon, r, (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m))$ is admissible. From the constructive proof of Theorem 3.2

it follows that the quotient manifold corresponding to N_ρ is a Seifert manifold with prescribed signature.

Example 4.2. We construct a series of admissible necklaces whose groups uniformize $SL_2(R)$ Seifert manifolds with exactly one singular fiber.

Consider the equilateral plane closed 32-gonal line L drawn in Figure 1. (In every square we have a quarter of the boundary of a regular 16-gon.)

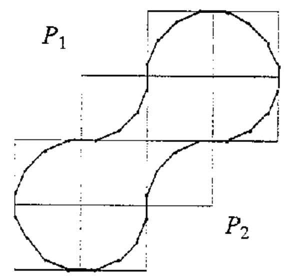

Figure 1.

Consider 32 round balls of radius R whose centers are vertices of the 32-gonal line, where R is chosen in such a way that every ball meets exactly two neighboring balls at the external angle $\pi/16\alpha$, α is an integer, $\alpha \geq 2$. The union of these balls forms a necklace N.

We pair sides with centers in the same square as it is drawn in Figure 2; the pairing transformations mapping the first side onto the second one is the composition of the inversion in the first sphere and the reflection transforming the first side to the second one.

The edges of the equipped necklace N form a cycle; a cyclic transformation of an edge is an elliptic transformation of order α pointwise preserving the edge. Therefore, the group of the necklace N is a quasi-Fuchsian group, for which the conecklace of N is a fundamental domain.

We deform the necklace N in the following way. Consider the plane P of L as contained in R^3. The axis a divides P into half-planes P_1 and P_2 (see Figure 1). Let L_i be the part of L contained in P_i. Rotate the halfplane P_2 about the axis a by $\pi\beta/\alpha$, where β is an integer, $|\beta| < \alpha$. Denote by L_2' the image of L_2 under the rotation.

We say that the broken line $L' = L_1 \cup L_2'$ is the result of *folding L along axis a by angle $\pi\beta/\alpha$*.

We can thread L' with 32 balls exactly as L; for $|\beta/\alpha|$ sufficiently small (e.g. for $|\beta/\alpha| \leq 2/3$) we have an equipped necklace N'.

The edges of N' form a cycle; a cyclic transformation of an edge is an elliptic transformation of order α which acts on the edge as a rotation by $2\pi\beta/\alpha$; the sum of angles at the edges of the cycle is equal to $2\pi/\alpha$.

The necklace N' is admissible; so, by Theorem 3.2, the corresponding quotient manifold M' is some $SL_2(R)$ Seifert manifold with exactly one singular

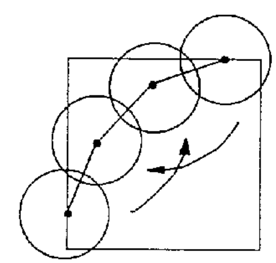

Figure 2.

fiber. The signature of this Seifert manifold is $(\mathfrak{D}, 16, [\beta/\alpha], (\alpha, \alpha\{\beta/\alpha\}))$ and the Euler number e is $-\beta/\alpha$.

§5. UNIFORMIZATION OF SEIFERT MANIFOLDS

This chapter contains the main results of the paper: Theorems 5.1, 5.2, and 5.3. In these theorems we develop and generalize the idea of Example 4.2.

Theorem 5.1. *Let M be a Seifert manifold with exactly one singular fiber. If $r = 6, 8$ or 10 and $0 < |e| \le (r-1)/7$ then M admits a uniformizable conformally flat structure.*

Proof. We shall prove the theorem for $r = 6, 8, 10$ by the same method. We construct three series of admissible necklaces.

(A) Construction of broken lines.

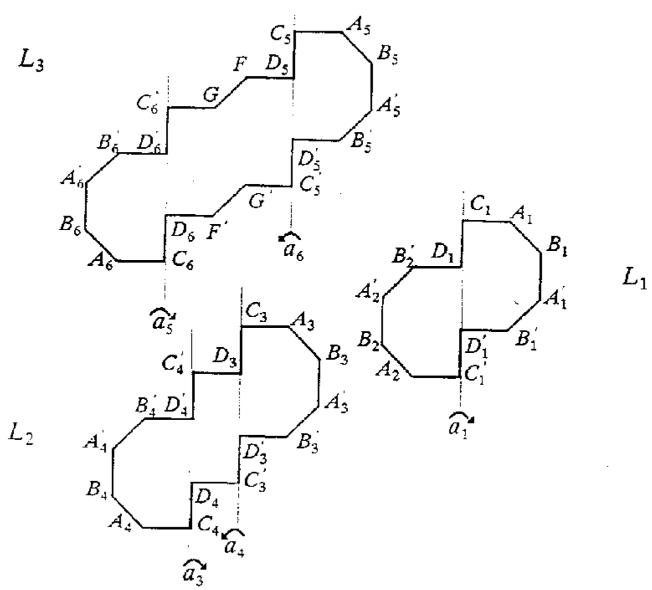

Figure 3.

Consider the plane closed equilateral $(4k+8)$-gonal lines L_k drawn in Figure 3, $k = 1, 2, 3$; in these broken lines the angles at every vertex are $\pi/2$ or $3\pi/4$. For $|\varphi| < \pi$, denote by L_1^{φ} the result of folding L_1 along axis a_1 by angle φ and by L_k^{φ} the

result of folding L_k along axes a_{2k-1} and a_{2k} by angle φ in opposite directions, $k = 2, 3$. (In Figure 4 you see the projection of L_k^{φ} onto a plane orthogonal to the axes.)

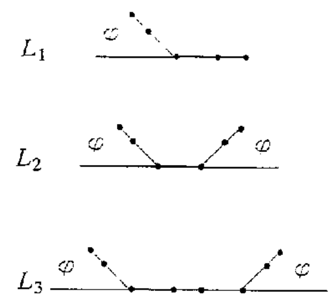

Figure 4.

(B) Construction of equipped necklaces. Consider $4k+8$ round balls of radius ρ whose centers are vertices of L_k^{φ}, where ρ is chosen in such a way that every ball meets exactly two neighbor balls at the external angle δ, $0 < \delta \le \pi/6$. Denote the union of these balls by $N_k(\delta, \varphi)$. For $|\varphi|$ sufficiently small, this is a necklace; suppose $N_k(\delta, \varphi)$ is a necklace for $|\varphi| < \varphi_k(\delta)$, and it is not a necklace for $|\varphi| = \varphi_k(\delta)$.

Let $\varphi_k = \inf\{\varphi_k(\delta) : 0 < \delta \le \pi/6\}$. One can show that

$$\varphi_1 > 3\pi/4, \ \varphi_2 > \pi/2, \ \varphi_3 > 2\pi/3.$$

So, for $0 < \delta \le \pi/6$, $|\varphi| < \varphi_k$, $N_k(\delta, \varphi)$ is a necklace.

Fix such φ and δ. Equip $N_k(\delta, \varphi)$. For $\varepsilon = \mathfrak{D}$ and $\mathfrak{N}$, it will be done differently; the equipped necklace will be denote by $N_k^{\varepsilon}(\delta, \varphi)$.

We shall denote a side of the necklace $N_k(\delta, \varphi)$ and the sphere containing this side by the letter for its center.

First we construct $N_k^{\mathfrak{D}}(\delta, \varphi)$.

We pair the sides A_i with A_i' by the transformation that is the composition of the inversion in the sphere A_i and the reflection mapping the side A_i onto the side A_i'. Analogously, we pair B_i with B_i'.

We pair C_i and C_i' by the transformation that is the composition of the following four transformations:

(1) the inversion in the sphere C_i;

(2) the reflection in the plane containing the vertex C_i and orthogonal to the bisectrix of the angle at the vertex C_i of L_k^{φ};

(3) the translation sending C_i to C_i';

(4) the rotation about the axis O_i mapping the image of the side C_i under the composition of the transformations 1, 2, 3 onto the side C_i'.

The sides D_i and D_i' are paired analogously.

We pair the sides F and F' by the transformation that is the composition of the three transformations 1, 2, 3 as above but with F, F' substituted for C_i, C_i'.

Analogously we pair G and G'.

Now we construct $N_k^{\mathfrak{n}}(\delta,\varphi)$. The equipment of this necklace differs from the one of $N_k^{\mathfrak{p}}(\delta,\varphi)$ by the pairing transformations for the sides $A_{2k}, A'_{2k}, B_{2k}, B'_{2k}$ only; here A_{2k} is paired with B_{2k} and A'_{2k} is paired with B'_{2k}.

We pair the sides A_{2k} with B_{2k} by the transformation that is the composition of the following three transformations:

(1) the inversion in the sphere A_{2k};

(2) the reflection mapping the side A_{2k} onto the side B_{2k};

(3) the rotation by π about the bisectrix of the angle at the vertex B_{2k} of L_k^{φ}.

The sides A'_{2k} and B'_{2k} are paired analogously. The edges of the equipped necklace $N_k^{\varepsilon}(\delta,\varphi)$ form a cycle, the sum of angles at the edges is equal to $(4k+8)\delta$, and cycle transformations act on edges as rotations by $2\varphi m_k$, where m_k is the number of the axes of the folding of the broken line L_k, that is, $m_1 = 1$, $m_2 = m_3 = 2$.

(C) For coprime integers α, β, $\alpha \geq 2$, $|\pi\beta/\alpha m_k| < \varphi_k$, put $\delta = \pi/\alpha(2k+4)$, $\varphi = \pi\beta/\alpha m_k$. The corresponding necklace is denoted by $N_k^{\varepsilon}[\alpha,\beta]$; clearly, this necklace is admissible. By Theorem 3.2, the quotient manifold arising from this necklace is a conformally flat Seifert manifold. It can be seen from the constructive proof of this theorem that the signature of this manifold is $(\varepsilon,\ 2k+4,\ [\beta/\alpha],\ (\alpha, \alpha\{\beta/\alpha\}))$. In particular, the Euler number $e = -[\beta/\alpha] - \{\beta/\alpha\} = -\beta/\alpha$.

(D) Now we are ready to complete the proof. Consider a Seifert manifold M with signature $(\varepsilon', r', b', (\alpha',\beta'))$, where $0 < |e'| \leq (r'-1)/7$, $r' = 6, 8$ or 10.

Let $k = (r'-4)/2$, $\alpha = \alpha'$, $\beta = \alpha' b' + \beta'$, $\varepsilon = \varepsilon'$. Note that $|\pi\beta/\alpha m_k| < \varphi_k$; indeed,

$$|\beta/\alpha| = |b' + \beta'/\alpha'| = |e'| \leq (r'-1)/7 = (2k+3)/7 < \varphi_k m_k/\pi .$$

The last estimate follows from the lower bound for φ_k given above.

By (C), the group of the necklace $N_k^{\varepsilon}[\alpha,\beta]$ uniformizes M. The proof is complete.

Theorem 5.2. *Let M be a Seifert manifold with at least one singular fiber. If $r \geq 6$ and $0 < |e| \leq (r-1)/8$ then M admits a conformally flat structure.*

Proof. Let the signature of M be

$$(\varepsilon, r, b, (\alpha_1,\beta_1), \ldots, (\alpha_m,\beta_m)).$$

To uniformize M, we construct an admissible necklace by copasting it from the necklaces described above. The first element of construction is the necklace N_R appearing in Example 4.1 for the signature

$$(\mathfrak{n}, p, (\alpha_1,\beta_1), \ldots, (\alpha_m,\beta_m))$$

where $p = t$ if r is even and $p = t+1$ otherwise, $t = [r/6]$. This necklace has m cycles of edges of length 1 and one "long" cycle. The module $M(R)$ of the jth side does not depend on $j = 1, \ldots, 2t$ and is a monotonic continuous function of R, and $\tau(R)$, the sum of angles at the edges in the long cycle, is a continuous function of R. Note that $M(R)$ tends to ∞ and $\tau(R)$ tends to 0 as R tends to 1; $M(R)$ tends to 0 as R tends to 0.

Modify the equipment of N_R as follows. We change $g_1, \ldots, g_{2t}$ putting, for $j = 2, 4, \ldots, 2t$, $g_j = R_j \circ I_j$, $g_{j-1} = g_j^{-1}$. Denote the modified equipped necklace by N_R^*.

We also need for our construction the equipped necklace $N_k^{\varepsilon}(\delta,\varphi)$ of Theorem 5.1. Note that $m(\delta)$, the module of the side A_{2k-1}, does not depend on k and is a continuous function of δ; $m(\delta)$ tends to ∞ as δ tends to 0. Denote $R(\delta) = M^{-1}(m(\delta))$, $\delta \in (0, \pi/6]$; clearly, the function $R(\delta)$ is continuous and $R(\delta)$ tends to 1 as δ tends to 0. Put $k = 1$ if $r \equiv 0, 1 \pmod 6$, $k = 2$ if $r \equiv 2, 3 \pmod 6$, $k = 3$ if $r \equiv 4, 5 \pmod 6$.

By the choice of $R(\delta)$, the side A_{2k-1} of $N_k^{\varepsilon}(\delta,\varphi)$ is Möbius equivalent to $(2d)$th side of $N^*_{R(\delta)}$, $d = 1, \ldots, t$. Copaste $N_k^{\varepsilon}(\delta,\varphi)$ and $N^*_{R(\delta)}$ along the side A_{2k-1} and the second side of $N^*_{R(\delta)}$; denote the result by N^1. Construct $N^2, \ldots, N^t$ as follows: N^d is copasted from N^{d-1} and $N_1^{\varepsilon}(\delta,\varphi)$ along the $(2d)$th side of $N^*_{R(\delta)}$ and A_1, $d = 2, \ldots, t$. The resulting necklace depends on δ, φ, ψ.

It can be shown that the equipped necklace N^t has m cycles of edges of length 1 (arising from one-element cycles of $N_{R(\delta)}$) and exactly one "long" cycle.

The sum of the angles at the edges of the long cycle can be computed as follows:

$$\Sigma(\delta) = 12(t-1)\delta + (4k+8)\delta + \tau(R(\delta)).$$

Every cyclic transformation of an edge in the long cycle is conjugate to a Möbius transformation that acts on the edge as a rotation by

$$\Theta(\psi,\varphi) = 2(t-1)\psi + 2m_k\varphi + 2\pi \sum_{i=1}^{m} \beta_i/\alpha_i ,$$

where m_k is the number of axes of the folding of broken line L_k. So, if $\Sigma(\delta) = 2\pi$ and $\Theta(\psi,\varphi) = -2\pi b$, the equipped necklace N^t is admissible and the corresponding quotient manifold is a Seifert manifold with prescribed signature. Therefore, to prove the theorem it suffices to find δ, ψ and φ such that $\Sigma(\delta) = 2\pi$ and $\Theta(\psi,\varphi) = -2\pi b$.

Clearly, $\Sigma(\delta)$ tends to 0 as δ tends to 0, and $\Sigma(\pi/6) > 12t\pi/6 \geq 2\pi$. Therefore, $\Sigma(\delta) = 2\pi$, for some $\delta \in (0, \pi/6)$.

The condition $\Theta(\psi,\varphi) = -2\pi b$ is equivalent to

$$2\pi e = 2(t-1)\psi + 2m_k\varphi .$$

For every A with $|A| < 2(t-1)\varphi_1 + 2m_k\varphi_k$ there exist ψ, φ such that $|\psi| < \varphi_1$, $|\varphi| < \varphi_k$, $2(t-1)\psi + 2m_k\varphi = A$.

Therefore, to find ψ, φ with $\Theta(\psi,\varphi) = -2\pi b$ it suffices to show that $|2\pi e| < 2(t-1)\varphi_1 + 2m_k\varphi_k$.

By the condition of the theorem, $|e| \leq (r-1)/8$. So we need to show that $(r-1)/8 < ((t-1)\varphi_1 + m_k\varphi_k)/\pi$.

Taking into account the inequalities $\varphi_1 > 3\pi/4$, $\varphi_2 > \pi/2$, $\varphi_3 > 2\pi/3$ (see the proof of Theorem 5.1, (B)), we see that $m_k\varphi_k/\pi > C_k$, where $C_1 = 3/4$, $C_2 = 1$, $C_3 = 4/3$.

So we need to prove the inequality $(r-1)/8 \leq 3(t-1)/4 + C_k$; it follows from the table.

r	k	$(r-1)/8$	$3(t-1)/4+C_k$
$6t$	1	$(6t-1)/8$	$3t/4$
$6t+1$	1	$6t/8$	$3t/4$
$6t+2$	2	$(6t+1)/8$	$3(t+1)/4$
$6t+3$	2	$(6t+2)/8$	$3(t+1)/4$
$6t+4$	3	$(6t+3)/8$	$3(t-1)/4+4/3$
$6t+5$	3	$(6t+4)/8$	$3(t-1)/4+4/3$

The proof of Theorem 5.2 is complete.

Theorem 5.3. *Let M be a Seifert manifold without singular fibers. If $r \geq 8$ and $0 < |e| \leq [r/8]$, then M admits a uniformizable conformally flat structure.*

Proof. Let the signature of M be (ε, r, b).

First consider the case $r = 8$. The Seifert manifold with signature $(\varepsilon, 8, \pm 1)$ can be uniformized by the group of the necklace $N_2^\varepsilon(\pi/8, \pm\pi/2)$ of Theorem 5.1.

Now suppose $r > 8$. As in Theorem 5.2, to uniformize M we produce an admissible necklace by copasting it from the necklaces constructed above.

The first element of construction is the necklace N_R described in Example 4.1 for the signature $(\mathfrak{N}, t+d)$, where $r = 8t+d$, $0 \leq d < 8$.

Modify the equipment of N_R as follows.

Denote $q = d$ if d is even and $q = d-1$ otherwise. Put $\sigma = \sigma_1\sigma_2$, where, for $n = 2(t+d-q)$,

$$\sigma_1 = (12)\dots(n-1,n)$$

and

$$\sigma_2 = (n+1,n+3)(n+2,n+4)\dots(n+2q-3,n+2q-1)(n+2q-2,n+2q).$$

We change $g_1,\dots,g_{2t},g_{n+1},\dots,g_{n+2q}$ by putting $g_j = R_j \circ I_j$, for $j = 2,4,\dots,2t,n+1,n+2,n+5,n+6,\dots,n+2q-3,n+2q-2$. (The remaining g_j's are defined as $g_j = g_{\sigma(j)}^{-1}$.) Denote the modified equipped necklace by N_R^*.

We also need for our construction the equipped necklaces $N_2^\varepsilon(\delta,\varphi)$ of Theorem 5.1. As in Theorem 5.2, let $m(\delta)$ be the module of the side A_3; $M(R)$, the module of an arbitrary side of the necklace N_R^*; $\tau(R)$, the sum of angles at the edges of N_R^* and $R(\delta) = M^{-1}(m(\delta))$, for $\delta \in (0,\pi/6]$.

By the choice of $R(\delta)$, the side A_3 of $N_2^\varepsilon(\delta,\varphi)$ is Möbius equivalent to the jth side of N_R^*, $j = 2,4,\dots,2t$. Construct $N^0,\dots,N^t$ as follows. Put $N^0 = N_R^*(\delta)$; N^d is copasted from N^{d-1} and $N_2^\varepsilon(\delta,\varphi)$ along the $(2d)$th side of $N_R^*(\delta)$ and A_3, $d = 1,\dots,t$.

It can be shown that the equipped necklace N^t has exactly one cycle of edges. The sum of angles at its edges can be computed as follows:

$$\Sigma(\delta) = 16t\delta + \tau(R(\delta)).$$

Every cyclic transformation of any edge is conjugate to the Möbius transformation that acts on it as a rotation by $4t\varphi$.

So if $\Sigma(\delta) = 2\pi$ and $4t\varphi = 2\pi b$, the equipped necklace N^t is admissible and the corresponding quotient manifold is a Seifert manifold with prescribed signature. Therefore, to prove the theorem it suffices to find δ and φ satisfying these equalities.

Clearly, $\Sigma(\delta)$ tends to 0 as δ tends to 0, and $\Sigma(\pi/6) > 16\pi/6 > 2\pi$. Therefore, $\Sigma(\delta) = 2\pi$, for some $\delta \in (0,\pi/6)$.

We need to show that, for the unique φ satisfying $4t\varphi = 2\pi b$, the condition $|\varphi| < \varphi_2$ holds. Taking into account the fact that $|e| \leq [r/8]$, we have

$$|\varphi| = |\pi b/2t| = |\pi e/2t| \leq \pi/2 < \varphi_2,$$

and we are done.

5.4. The hyperbolic orbifolds H^4/G arising from the constructions of this section are isomorphic to certain plane bundles over 2-orbifolds whose unit circle bundles are Seifert manifolds.

In particular, we have

Corollary 5.4. *Let L be the total space of a plane bundle over a closed surface X. Let e be the Euler number of this plane bundle. Suppose $\chi(X) \leq -6$, $|e| \leq [(2-\chi(X))/8]$. Then L admits a complete hyperbolic structure.*

Remark 5.5. The maximal value of $|e/\chi|$ for conformally flat Seifert manifolds arising from our constructions is 1/6 (for the signature $(\varepsilon, 8, 1)$). It seems plausible that using Kapovich's machinery [3, §2, 3.2–3.4] one can prove Theorem 5.3 for $r \geq 8$, $0 < |e| \leq (r-2)/6$.

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LARGE DEVIATIONS AND TESTING STATISTICAL HYPOTHESES. III. ASYMPTOTICALLY OPTIMAL TESTS FOR COMPOSITE HYPOTHESES

A.A.Borovkov and A.A.Mogulskii

Abstract

In Part III of this article, the results of Parts I and II are employed to construct, on the basis of concepts from the theory of large deviations, asymptotically optimal criteria for testing a composite hypothesis against a composite aternative.

Key words and phrases: maximum likelihood ratio tests, maximum likelihood tests, MLRT, MLT, MLR test, ML test, Bayesian approach, minimax approach.

§8. THE STATEMENT OF THE PROBLEM. SOME CLASSES OF ASYMPTOTICALLY OPTIMAL TESTS

1. *The statement of the problem.* Let $(\mathscr{X}, \mathscr{B})$ be a measurable space and Θ an arbitrary (the so-called parametric) set. Consider a family of probability measures

$$\mathscr{P} = \{ \mathbf{P}_\theta, \theta \in \Theta \}$$

defined on the σ-algebra $\mathscr{B}$ and depending on the parameter $\theta \in \Theta$.

Let Θ_1 and Θ_2 be two given nonempty disjoint subsets of the set Θ. It is necessary to select one of the two hypotheses

$$H_1 = \{ \theta \in \Theta_1 \} \text{ and } H_2 = \{ \theta \in \Theta_2 \}$$

using the observations which constitute the sample

$$X_n = (x_1, \dots, x_n), \; x_i \in \mathrm{P}_\theta .$$

The collection

$$E_n = (\Theta_1, \Theta_2, \mathscr{P}, X_n),$$

which we call a statistical experiment for testing two composite statistical hypotheses (or simply an experiment), provides a complete specification of the problem formulated above.

A decision rule (or a test) T_n for the problem of testing the hypotheses H_1 and H_2 is defined by a measurable mapping $T_n = T_n(X_n) : \mathscr{X}^n \to [0, 1]$. The value $T_n(X_n)$ of the function T_n at the point X_n is interpreted as the probability of accepting the

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