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Rotationally Symmetric Planes in Comparison Geometry

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Ph.D., Emory University, 2012

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An abstract of
A dissertation submitted to the Faculty of the Graduate School
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Abstract

Rotationally Symmetric Planes in Comparison Geometry
By Eric Choi

Kondo-Tanaka generalized the Toponogov Comparison Theorem so that an arbitrary noncompact manifold M can be compared with a rotationally symmetric plane M_m (defined by the metric $dr^2 + m^2(r)d\theta^2$), and they used this to show that if M_m satisfies certain conditions, then M must be topologically finite. We substitute one of the conditions for M_m with a weaker condition and show that our method using this weaker condition enables us to draw further conclusions on the topology of M. We also completely remove one of the conditions required for the Sector Theorem, another important result by Kondo-Tanaka. Cheeger-Gromoll showed that if M has nonnegative sectional curvature, then M contains a boundaryless, totally convex, compact submanifold S, called a *soul*, such that M is homeomorphic to the normal bundle over S. We show that in the case of a rotationally symmetric plane M_m, the set of souls is a closed geometric ball centered at the origin, and if furthermore M_m is a von Mangoldt plane, then the radius of this ball can be explicitly determined. We prove that the set of critical points of infinity in M_m is equal to this set of souls, and we make observations on the set of critical points of infinity when M_m is von Mangoldt with negative sectional curvature near infinity. Finally, we set out conditions under which M_m can be guaranteed an annulus free of critical points of infinity and show that we can construct a von Mangoldt plane M_m that is a cone near infinity and for which $m'(r)$ near infinity is prescribed to be any number in $(0, 1]$.

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to Evelyn

Contents

Chapter 1

Introduction

We give below two different versions of our introduction: short and long. As the terms suggest, the short version is tailored to give non-geometers a bird's-eye view of the overarching themes and salient theorems. The long version gives geometers a more technical preparation for reading the thesis. The long version is self-contained, so if you wish to read it, you can skip the short version. In the final section, we give a quick overview of the structure of this thesis.

1.1 Short Introduction

Global Riemannian geometry seeks to relate geometric data to topological data. It is often of particular interest if we can show that a certain set of traits imply that a noncompact manifold M is topologically finite, i.e. that it is homeomorphic to the interior of a compact set with boundary. According to the critical point theory of distance functions [Gro93], [Gre97, Lemma 3.1], [Pet06, Section 11.1], M is topologically finite if the set of critical points of the distance function to some point $p \in M$, denoted $d(\cdot, p)$, is bounded; we say that $q \in M$ is a *critical point of* $d(\cdot, p)$ if for every $v \in T_qM$ there exists a minimal geodesic γ joining q to p such that $\measuredangle(\dot{\gamma}(0), v) \leq \frac{\pi}{2}$.

In chapter 8, we discuss results in [KT10], which applies the above principle, and we improve on them. The authors generalize the Toponogov Comparison Theorem to show that if the radial sectional curvature of M from a basepoint p is bounded below by that of a rotationally symmetric plane M_m with finite total curvature and a sector free of cut points, then M must be topologically finite. (We define a rotationally symmetric plane M_m as $\mathbb{R}^2$ together with metric $dr^2 + m^2(r)d\theta^2$), where $m : (0, \infty) \to (0, \infty)$, $m(0) = 0$, $m'(0) = 1$, is smooth and extends to a smooth odd function around the origin. Examples of rotationally symmetric planes are hyperboloids and paraboloids.) We improve on this result by substituting the condition of total curvature (of M_m) with the weaker condition of $\sup\{m'(r)\} < \infty$. We also show that if $\sup\{m'\} = 1$, if M_m is not isometric to $\mathbb{R}^2$ (with the standard Euclidean metric), and if basepoint $p \in M$ is a critical point of infinity, then M is homeomorphic to $\mathbb{R}^n$. (See below in this introduction for a definition of a *critical point of infinity*.)

We also improve on the Sector Theorem in [KT10] in chapter 8: If M_m is von Mangoldt or Cartan-Hadamard outside a compact set and has finite total curvature, then it must have a sector free of cut points. The authors feel that the Sector Theorem "clarifies the real significance of finite total curvature and the validity of the Main Theorem [in the previous paragraph]." We improve on the Sector Theorem by showing that the condition of finite total curvature can be dropped entirely.

If the sectional curvature of M is everywhere nonnegative, then the set of critical points of $d(\cdot, p)$ must be bounded, so M must be topologically finite. In fact we know much more: According to the Soul Theorem by Cheeger-Gromoll (discussed in chapter 3), not only is M topologically finite, but there exists a compact, totally convex, boundaryless submanifold

S, called a soul, such that M is diffeomorphic to the normal bundle over S. For example, any soul of a contractible space such as $\mathbb{R}^n$ is isometric to a point, and a soul of the infinite cylinder $\mathbb{R} \times S^1$ is isometric to S^1. The existence of a totally convex submanifold is in itself remarkable in view of the fact that most Riemannian manifolds do not even contain a nontrivial totally geodesic submanifolds. All souls of M are isometric to each other, and any submanifold $S \subset M$ isometric to a soul is called a *pseudo-soul*. As the term suggests, S does not qualify as a soul just because it is isometric to S; for S to be a soul, it must be the end result of the soul construction procedure. So even if we understand the geometry of S, it is still natural to wonder which submanifolds isometric to S are actually souls of M.

Another distinguished set of points that may be found in a noncompact manifold is the set of critical points of infinity. A point $q \in M$ is a *critical point of infinity* if for every $v \in T_qM$ there exists a ray γ emanating from $q = \gamma(0)$ such that $\measuredangle(\dot{\gamma}(0), v) \leq \frac{\pi}{2}$. While the concept of souls applies only to manifolds of everywhere nonnegative sectional curvature, such a curvature restriction is not needed for critical points of infinity.

In the case of M_m with $G_m \geq 0$, since M_m is diffeomorphic to $\mathbb{R}^2$, we know a priori that any soul of M_m is isometric to a point. But in chapters 5 and 6, we show that the set of souls equals the set of critical point of infinity and that this set is a closed metric ball centered at the origin. If furthermore M_m is von Mangoldt, then the radius of this ball can be explicitly determined. Also in chapter 5, we present our observations on the set of critical points of infinity when the sectional curvature of M_m is not everywhere nonnegative, and we also show that certain conditions on m' guarantee an annulus in M_m free of critical points of infinity. Finally, in chapter 7, we show that we can construct a von Mangoldt plane that is a cone near infinity with $m'(r)$ prescribed.

1.2 Long Introduction

Global Riemannian geometry seeks to relate geometric data to topological data. It is often of particular interest if we can show that a certain set of traits imply that a manifold is topologically finite, i.e. that it is homeomorphic to the interior of a compact manifold with boundary.

Let M denote a complete noncompact Riemannian manifold; let M_m denote a *rotationally symmetric plane*, defined as $\mathbb{R}^2$ equipped with a smooth, complete, rotationally symmetric Riemannian metric given in polar coordinates as $g_m := dr^2 + m^2(r)d\theta^2$; and let o denote the origin in $\mathbb{R}^2$. In [KT10], the authors generalize the Toponogov Comparison Theorem to show that if the radial sectional curvature of M from basepoint p is bounded below by that of a plane M_m with finite total curvature and a sector free of cut points, then M is topologically finite.

By the critical point theory of distance functions developed by Grove-Shiohama [Gro93], [Gre97, Lemma 3.1], [Pet06, Section 11.1], topological finiteness of M would follow once it is shown that the set of critical points of $d(\cdot, p)$, the distance function to p, is bounded for some $p \in M$.

In Theorem 8.4.6 below, we show that finiteness of total curvature in the above mentioned result of Kondo-Tanaka can be replaced with a weaker assumption as follows. Set

$$N := \sup\{m'(r)\} \quad \text{and} \quad V(\delta) := \{q \in M_m \mid 0 < \theta(q) < \delta\}.$$

Theorem 8.4.6. *Let the radial curvature of (M, p) be bounded below by that of M_m with $N < \infty$ and a sector $V(\delta)$ free of cut points. Then M is topologically finite.*

In Theorem 8.4.7 below, a point q in a Riemannian manifold is called a *critical point of infinity* if each unit tangent vector at q makes angle $\leq \frac{\pi}{2}$ with a ray that starts at q; a geodesic $\gamma : [0, \infty) \to M$ is a *ray* if the image of $\gamma|_{[0,s]}$ is distance-minimizing for every $s \in [0, \infty)$. Also, let N be as in Theorem 8.4.6.

Theorem 8.4.7. *Let the radial curvature of* (M, p) *be bounded below by that of* M_m *with a cut-point-free sector* $V(\delta)$. *Suppose:*

1) $N = 1$

2) M_m *is not isometric to* $\mathbb{R}^2$

3) $\delta > \frac{\pi}{2}$

Then if p *is a critical point of infinity,* M *is homeomorphic to* $\mathbb{R}^n$, *where* n *is the dimension of* M.

Since the generalized Toponogov Theorem in [KT10] requires that M_m have a sector free of cut points, it is natural wonder what types of rotationally symmetric planes have this property. One of the main results of [KT10] is the Sector Theorem, stated below.

Theorem 8.5.14. (Sector Theorem) *Let* M_m *be a noncompact rotationally symmetric plane that is von Mangoldt or Cartan-Hadamard outside a ball of finite radius* $R > 0$ *about* o. *Also assume* M_m *has a finite total curvature. Then* M_m *has a sector free of cut points.*

Remark 1.2.1. In [KT10], the authors introduce the Sector Theorem with the comment that it "clarifies the real significance of finite total curvature and the validity of the Main Theorem (of [KT10])." However, in our thesis, we show that the condition of finite total curvature in the Sector Theorem can be dropped altogether.

The set of critical points of infinity of M_m, denoted $\mathfrak{C}_m$, is of interest;

the following corollary of the generalized Toponogov Comparison Theorem gives one reason to study $\mathfrak{C}_m$.

Proposition 8.4.10. *Let M be a complete noncompact Riemannian manifold with radial curvature bounded below by the curvature of a von Mangoldt plane M_m, and let r, r_m denote the distance functions to the basepoints p, o of M, M_m, respectively. If q is a critical point of r, then $r(q)$ is contained in $r_m(\mathfrak{C}_m)$.*

Combined with the critical point theory of distance functions [Gro93], [Gre97, Lemma 3.1], [Pet06, Section 11.1], Proposition 8.4.10 implies the following.

Proposition 1.2.2. *In the setting of Lemma 8.4.10, for any c in $[a,b] \subset r_m(M_m - \mathfrak{C}_m)$,*

- *the r^{-1}-preimage of $[a,b]$ is homeomorphic to $r^{-1}(a) \times [a,b]$, and the r^{-1}-preimages of points in $[a,b]$ are all homeomorphic;*
- *the r^{-1}-preimage of $[0,c]$ is homeomorphic to a compact smooth manifold with boundary, and the homeomorphism maps $r^{-1}(c)$ onto the boundary;*
- *if $K \subset M$ is a compact smooth submanifold, possibly with boundary, such that $r(K) \supset r_m(\mathfrak{C}_m)$, then M is diffeomorphic to the normal bundle of K.*

If M_m is von Mangoldt and $G_m(0) \leq 0$, then $G_m \leq 0$ everywhere, so every point is a *pole*, defined as a point from which there is a ray emanating in every possible direction. Hence $\mathfrak{C}_m = M_m$, so that Lemma 8.4.10 yields no information about the critical points of r. Of course, there are other ways to get this information as illustrated by classical Gromov's estimate: if M_m is the standard $\mathbb{R}^2$, then the set of critical points of r is compact; see e.g. [Gre97, page 109].

Given a complete noncompact manifold M that is topologically finite, can we estimate the radius of the subset $K \subset M$ that determines the topology of M? In particular, can the radius of $\mathfrak{C}_m$ be determined? Theorem 5.1.1 below gives what we understand about $\mathfrak{C}_m$ when M_m has nonnegative sectional curvature, and parts (iv) and (v) provide a way of bounding and determining the radius of $\mathfrak{C}_m$ given that M_m also is von Mangoldt.

Theorem 5.1.1. *Given M_m, suppose $G_m \geq 0$. Then*
(i) C_m is a closed R_m- ball centered at o for some $R_m \in [0, \infty]$.
(ii) R_m is positive if and only if $\int_1^\infty m^{-2}$ is finite.
(iii) R_m is finite if and only if $m'(\infty) < \frac{1}{2}$.
(iv) If M_m is von Mangoldt and R_m is finite, then the equation $m'(r) = \frac{1}{2}$ has a unique solution ρ_m, and the solution satisfies $\rho_m > R_m$ and $G_m(\rho_m) > 0$.
(v) If M_m is von Mangoldt and R_m is finite and positive, then R_m is the unique solution of the integral equation $\int_x^\infty \frac{m(x)dr}{m(r)\sqrt{m^2(r)-m^2(x)}} = \pi$.

Combining Proposition 8.4.10, Proposition 1.2.2, and Theorem 5.1.1, we have the following simple estimate:

Proposition 1.2.3. *Let M be a complete noncompact Riemannian manifold with radial curvature from the basepoint p bounded below by the curvature of a von Mangoldt plane M_m. If $G_m \geq 0$ and $m'(\infty) < \frac{1}{2}$, then M is homeomorphic to the metric ρ_m-ball centered at p, where ρ_m is the unique solution of $m'(r) = \frac{1}{2}$.*

Theorem 5.1.1 should be compared with the following results of Tanaka:

- the set of poles in any M_m is a closed metric ball centered at o of some radius R_p in $[0, \infty]$ [Tan92b, Lemma 1.1].

- $R_p > 0$ if and only if $\int_1^\infty m^{-2}$ is finite and $\liminf_{r\to\infty} m(r) > 0$ [Tan92a].

- if M_m is von Mangoldt, then R_p is a unique solution of an explicit integral equation [Tan92a, Theorem 2.1].

It is natural to wonder when the set of poles equals $\mathfrak{C}_m$, and we answer the question when M_m is von Mangoldt.

Theorem 5.2.1. *If M_m is a von Mangoldt plane, then*

(a) *If R_p is finite and positive, then the set of poles is a proper subset of the component of $\mathfrak{C}_m$ that contains o.*

(b) *$R_p = 0$ if and only if $\mathfrak{C}_m = \{o\}$.*

Of course $R_p = \infty$ implies $\mathfrak{C}_m = M_m$, but the converse is not true: Theorem 7.2.1 ensures the existence of a von Mangoldt plane with $m'(\infty) = \frac{1}{2}$ and $G_m \geq 0$, and for this plane $\mathfrak{C}_m = M_m$ by Theorem 5.1.1, while R_p is finite by Remark 6.0.5.

We say that a ray γ in M_m *points away from infinity* if γ and the segment $[\gamma(0), o]$ make an angle $< \frac{\pi}{2}$ at $\gamma(0)$. Define $A_m \subset M_m - \{o\}$ as follows: $q \in A_m$ if and only if there is a ray that starts at q and points away from infinity; by symmetry, $A_m \subset \mathfrak{C}_m$.

Theorem 5.2.2. *If M_m is a von Mangoldt plane, then A_m is open in M_m.*

Any plane M_m with $G_m \geq 0$ has another distinguished subset, namely the set of souls, i.e. submanifolds produced via the soul construction of Cheeger-Gromoll. In fact Cheeger-Gromoll showed that soul construction can be done on any complete noncompact manifold M with nonnegative sectional curvature to produce a soul, which is a compact, totally convex, boundaryless submanifold S such that M is diffeomorphic to the normal bundle over S. For example, a soul of any contractible space (such as any plane M_m) is isometric to a point, and a soul of the infinite cylinder $\mathbb{R}$ x S^1 is isometric to S^1. The existence of a totally convex submanifold is in itself remarkable in view of the fact that most Riemannian manifolds do not even contain nontrivial totally geodesic submanifolds ([ChEb], Preface).

All souls of any manifold M are isometric to each other. Any submanifold $S' \subset M$ isometric to a soul is called a *pseudo-soul*. As the term suggests, S' does not qualify as a soul just because it is isometric to S; for S' to be a soul, it must be the end result of the soul construction procedure. So even if we understand the geometry of S, it is still natural to wonder which submanifolds isometric to S are actually souls of M. We address this issue with respect to a rotationally symmetric plane M_m:

Theorem 6.0.1. *If $G_m \geq 0$, then $\mathfrak{C}_m$ is equal to the set of souls of M_m.*

The soul construction takes as input a basepoint $p \in M$, and if M is contractible and any soul S is therefore a point, the soul construction gives a continuous family of compact totally convex subsets that starts with S and ends with M, and according to [Men97, Proposition 3.7] $q \in M$ is a critical point of infinity if and only if there is a soul construction such that the associated continuous family of totally convex sets drops in dimension at q. In particular, any point of S is a critical point of infinity, which can also be seen directly; see the proof of [Mae75, Lemma 1]. In Theorem 6.0.1 we prove conversely that every point of $\mathfrak{C}_m$ is a soul; for this M_m need not be von Mangoldt.

In regard to part (iii) of Theorem 5.1.1, it is worth mentioning $G_m \geq 0$ implies that m' is non-increasing, so $m'(\infty)$ exists, and moreover, $m'(\infty) \in [0,1]$ because $m \geq 0$. As we note in Remark 7.1.5 for any von

Mangoldt plane M_m, the limit $m'(\infty)$ exists as a number in $[0,\infty]$. It follows that any M_m with $G_m \geq 0$ and any von Mangoldt plane M_m admits total curvature, which equals $2\pi(1 - m'(\infty))$ and hence takes values in $[-\infty, 2\pi]$; thus $m'(\infty) = \frac{1}{2}$ if and only if M_m has total curvature π. Standard examples of von Mangoldt planes of positive curvature are the one-parametric family of paraboloids, all satisfying $m'(\infty) = 0$ [SST03, Example 2.1.4], and the one-parametric family of two-sheeted hyperboloids parametrized by $m'(\infty)$, which takes every value in $(0,1)$ [SST03, Example 2.1.4].

A property of von Mangoldt planes, discovered in [Ele80, Tan92b] and crucial to our results, is that the cut locus of any $q \in M_m - \{o\}$ is a ray that lies on the meridian opposite q. (If M_m is not von Mangoldt, its cut locus is not fully understood, but it definitely can be disconnected [Tan92a, page 266], and known examples of cut loci of compact surfaces of revolution [GS79, ST06] suggest that it could be complicated).

As we note in Lemma 4.3.10, if M_m is a von Mangoldt plane, and if $q \neq o$, then $q \in \mathfrak{C}_m$ if and only if the geodesic tangent to the parallel through q is a ray. Combined with Clairaut's relation this gives the following "choking" obstruction for a point q to belong to $\mathfrak{C}_m$:

Lemma 4.3.11. *If M_m is von Mangoldt and $q \in \mathfrak{C}_m$, then $m'(r_q) > 0$ and $m(r) > m(r_q)$ for $r > r_q$, where r_q is the r-coordinate of q.*

We also show in Lemma 4.3.5 that if M_m is von Mangoldt and $\mathfrak{C}_m \neq o$, then there exists ρ such that $m(r)$ is increasing and unbounded on $[\rho, \infty)$.

The following theorem collects most of what we know about $\mathfrak{C}_m$ for a von Mangoldt plane M_m with some negative curvature, where the case $\liminf_{r\to\infty} m(r) = 0$ is excluded because then $\mathfrak{C}_m = \{o\}$ by Lemma 4.3.11.

Theorem 5.3.1. *If M_m is a von Mangoldt plane with a point where $G_m < 0$ and such that $\liminf_{r\to\infty} m(r) > 0$, then*

(1) M_m *contains a line and has total curvature* $-\infty$*;*

(2) *if* m' *has a zero, then neither* A_m *nor* $\mathfrak{C}_m$ *is connected;*

(3) $M_m - A_m$ *is a bounded subset of* M_m*;*

(4) *the ball of poles of* M_m *has positive radius.*

In Example 5.3.2 we construct a von Mangoldt plane M_m to which part (2) of Theorem 5.3.1 applies. In Example 5.3.3 we produce a von Mangoldt plane M_m such that neither A_m nor $\mathfrak{C}_m$ is connected while $m' > 0$ everywhere. We do not know whether there is a von Mangoldt plane such that $\mathfrak{C}_m$ has more than two connected components.

Because of Lemma 8.4.10 and Corollary 1.2.2, one is interested in subintervals of $(0, \infty)$ that are disjoint from $r(\mathfrak{C}_m)$, as e.g. happens for any interval on which $m' \leq 0$, or for the interval (R_m, ∞) in Theorem 5.1.1. To this end we prove the following result, which is a consequence of Theorem 5.4.2.

Theorem 5.4.3. *Let* M_n *be a von Mangoldt plane with* $G_n \geq 0$*,* $n(\infty) = \infty$*, and such that* $n'(x) < \frac{1}{2}$ *for some* x*. Then for any* $z > x$ *there exists* $y > z$ *such that if* M_m *is a von Mangoldt plane with* $n = m$ *on* $[0, y]$*, then* $r(\mathfrak{C}_m)$ *and* $[x, z]$ *are disjoint.*

In general, if M_m, M_n are von Mangoldt planes with $n = m$ on $[0, y]$, then the sets $\mathfrak{C}_m$, $\mathfrak{C}_n$ could be quite different. For instance, if M_n is a paraboloid, then $\mathfrak{C}_n = \{o\}$, but by Example 5.3.3 for any $y > 0$ there is a von Mangoldt M_m with some negative curvature such that $m = n$ on $[0, y]$, and by Theorem 5.3.1 the set $M_m - \mathfrak{C}_m$ is bounded and $\mathfrak{C}_m$ contains the ball of poles of positive radius.

In order to construct a von Mangoldt plane with prescribed G_m it suffices to check that 0 is the only zero of the solution of the Jacobi initial value problem (7.1.7) with $K = G_m$, where G_m is smooth on $[0, \infty)$. Prescribing values of m' is harder. It is straightforward to see that if M_m is a von Mangoldt plane such that m' is constant near infinity, then $G_m \geq 0$ everywhere and $m'(\infty) \in [0, 1]$. We do not know whether there is a von Mangoldt plane with $m' = 0$ near infinity, but all the other values in $(0, 1]$ can be prescribed:

Theorem 7.2.1. *For every $s \in (0, 1]$ there is $\rho > 0$ and a von Mangoldt plane M_m such that $m' = s$ on $[\rho, \infty)$.*

Thus each cone in $\mathbb{R}^3$ can be smoothed to a von Mangoldt plane, but we do not know how to construct a (smooth) capped cylinder that is von Mangoldt.

1.3 Structure of the Thesis

Basic definitions, concepts, and theorems are discussed in chapters 2 and 3. In particular, section 2.5.3 culminates in a much-used theorem by M. Tanaka, and chapter 3 outlines the proof of the Soul Theorem. From section 4.3 of chapter 4 on to the end of the thesis, most of the results are our own work. In chapter 4, sections 4.1 and 4.2, we discuss the Clairaut relation and the Turn Angle Formula, important tools for analyzing the behavior of geodesics in a rotationally symmetric plane, M_m. The rest of chapter 4 from section 4.3 on presents various lemmas on the behavior of geodesics in M_m, used to prove our results in chapters 5 and 6. Chapter 5 presents our results on the geometry and topology of the set of critical points infinity in M_m. In chapter 6 we show that the set of souls in M_m is equal to the set of critical points of infinity of M_m. In chapter 7 we discuss how we can prescribe the slope of $m(r)$ near infinity when M_m is von Mangoldt. Chapter 8 presents our improvements on results in [KT10], discussed above.

Chapter 2

Basic Facts and Definitions

We discuss ideas that are building blocks to our work. Especially central to our results is Theorem 2.5.23, which describes an important attribute of von Mangoldt planes. Many of the definitions and remarks in this chapter are closely modeled on expositions in [Car], [GrWal] [Lee], and [SST03].

Definition 2.0.1. Let M be a smooth manifold, let T_pM be the tangent space of a point $p \in M$, and let $\mathbf{x} : U \subset \mathbb{R}^n \to M$ be a system of coordinates around p, with $\mathbf{x}(x_1, x_2, ..., x_n) = q \in \mathbf{x}(U)$ and $\frac{\partial}{\partial x_i}(q) = d\mathbf{x}_q(0, ..., 1, ..., 0)$. A *Riemannian metric* on M is a correspondence that associates to T_pM an inner product $\langle \cdot, \cdot \rangle_p$ such that $\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \rangle_q = g_{ij}(x_1, ..., x_n)$ is a differentiable function on U.

Definition 2.0.2. M is a *Riemannian manifold* if it is a smooth manifold equipped with a Riemannian metric. We sometimes use the notation (M, g) to denote a smooth manifold M paired with a Riemannian metric g.

Definition 2.0.3. A smooth curve $\gamma : [a, b] \to M$ is a *geodesic* if, given any point p on γ, there exists an ϵ neighborhood of p on γ such that if x, y are in the neighborhood, the length of the subsegment of γ joining x and y is $\leq$ the length of every other curve joining x and y. This is equivalent to saying that γ is a geodesic if and only if $\nabla_{\dot{\gamma}}\dot{\gamma} \equiv 0$, where ∇ is the Riemannian connection associated with M.

A curve $\gamma : [a, b] \to M$ is a *minimal geodesic* if the length of γ is $\leq$ the length of every other curve on M joining $\gamma(a)$ and $\gamma(b)$; that is, the length of γ equals $d(\gamma(a), \gamma(b))$, where the distance function is derived from the Riemannian metric specific to M. We sometimes say that γ is *distance-minimizing* between $\gamma(a)$ and $\gamma(b)$.

Remark 2.0.4. As an example differentiating a non-minimal geodesic from a minimal geodesic, consider a sphere of radius R. The image of any complete geodesic in a sphere is a great circle (i.e. a circle of radius R), but only subarcs of length $\leq \pi R$ in the great circles are images of minimal geodesics; if any arc in a great circle exceeds length πR, then it will not minimize the distance between its endpoints.

Definition 2.0.5. A Riemannian manifold M is *complete* if, given any $p \in M$, any geodesic $\gamma(t)$ starting from p is defined for all values of the parameter $t \in \mathbb{R}$. Equivalently, M is complete if it is complete as a metric space. Completeness of M implies that given any $p, q \in M$, there exists a minimal geodesic joining p to q. As an example of a space that is not complete, consider $\mathbb{R}^2 \setminus \{0\}$. For any $t \in \mathbb{R}$, there does not exist a minimal geodesic joining $p = (t, t)$ to $q = (-t, -t)$.

Remark 2.0.6. Throughout this thesis, every Riemannian manifold M will be assumed to be complete and noncompact.

Definition 2.0.7. Given any point $q \in M$ and a geodesic γ emanating from $q = \gamma(0)$, we say that $q' = \gamma(s_0)$, $s_0 > 0$ is a *cut point* of q if γ is a minimal geodesic on $[0, s]$ for all $s \leq s_0$ but is not minimal for all $s > s_0$. The collection of all cut points of q is called the *cut locus* of q. If γ is the only geodesic connecting any $q, q' \in M$, then it must be minimal. On the other hand, if two minimal geodesics emanating from q meet at some $q' \neq q$, they are not minimal beyond q'.

Definition 2.0.8. A geodesic $\gamma : [0, \infty) \to M$ is a *ray* if, for every t_1, $t_2 \in [0, \infty)$, γ minimizes the distance between $\gamma(t_1)$ and $\gamma(t_2)$. A geodesic $\gamma : (-\infty, \infty) \to M$ is a *line* if, for every t_1, $t_2 \in (-\infty, \infty)$, γ minimizes the distance between $\gamma(t_1)$ and $\gamma(t_2)$.

Remark 2.0.9. Every point $p \in M$ (assumed to be noncompact and complete) has at least one ray emanating from it. Indeed, since M is noncompact, there exists a sequence of points $\{q_n\}$ such that $d(p, q_n) \to \infty$ as $n \to \infty$. Let γ_n be a minimal geodesic connecting p to q_n. The sequence $\{\gamma_n\}$ must subconverge to a geodesic γ, and γ must be a ray since the function

$$f : \{v \in T_q M; |v| = 1\} \to \mathbb{R}^+ \cup \{\infty\}, \quad v \mapsto \sup\{t > 0; \quad d(p, \exp(tv)) = t\}$$

is continuous.

Definition 2.0.10. Let M and N be Riemannian manifolds. We say that M and N are *isometric*, or that $\phi : M \to N$ is an *isometry,* if ϕ is a diffeomorphism and $\langle u, v \rangle_p = \langle d\phi(u), d\phi_p(v) \rangle_{\phi(p)}$ for all $p \in M$, $u, v \in T_p M$. In particular, the distance between any two points p, p' in M equals the distance between $\phi(p), \phi(p')$ in N. Loosely speaking, isometry means equivalence between two spaces to a geometer, even as isomorphism and homeomorphism mean equivalence between two spaces to an algebraist or a topologist, respectively.

Definition 2.0.11. Let M be a Riemannian manifold, p an arbitrary point in M, and γ an arbitrary geodesic passing through p. We define the *exponential function at p*, $\exp_p : T_p M \to M$, by $\exp_p(v) = \gamma(|v|)$, where $\frac{v}{|v|} = \dot{\gamma}(0)$.

Definition 2.0.12. A point $q \in M$ is a *critical point of infinity* if, given any $v \in T_qM$ (the tangent space of q), there exists a ray γ emanating from q such that $\measuredangle(\dot{\gamma}(0), v) \leq \frac{\pi}{2}$.

Definition 2.0.13. In a complete Riemannian manifold M, a point p is a *pole* if every geodesic emanating from p is a ray. Clearly the set of poles is a subset of the set of critical points of infinity in any manifold.

2.1 Notations and Conventions

All geodesics are parametrized by arclength. Minimal geodesics of finite length will sometimes be called *segments*. We will use M_m to denote a rotationally symmetric plane (see Section 2.4). Given $\mathbb{R}^2$, let ∂_r, ∂_θ denote the vector fields dual to dr, $d\theta$, and let o denote the origin. Given $q \neq o$, denote its polar coordinates by θ_q, r_q. Let γ_q, μ_q, τ_q denote the geodesics defined on $[0, \infty)$ that start at q in the direction of ∂_θ, ∂_r, $-\partial_r$, respectively. We refer to $\tau_q|_{(r_q,\infty)}$ as the *meridian opposite* q; note that $\tau_q(r_q) = o$. Also set $\kappa_{\gamma(s)} := \angle(\dot{\gamma}(s), \partial_r)$.

We write $\dot{r}$, $\dot{\theta}$, $\dot{\gamma}$, $\dot{\kappa}$ for the derivatives of $r_{\gamma(s)}$, $\theta_{\gamma(s)}$, $\gamma(s)$, $\kappa_{\gamma(s)}$ by s, while m' denotes $\frac{dm}{dr}$, and proceed similarly for higher derivatives.

Let $\hat{\kappa}(r_q)$ denote the maximum of the angles formed by μ_q and rays emanating from $q \neq o$; let ξ_q denote the ray with $\xi_q(0) = q$ for which the maximum is attained, i.e. such that $\kappa_{\xi_q(0)} = \hat{\kappa}(r_q)$.

A geodesic γ in $M_m - \{o\}$ is called *counterclockwise* if $\frac{d}{ds}\theta_{\gamma(s)} > 0$ and *clockwise* if $\frac{d}{ds}\theta_{\gamma(s)} < 0$ for some (or equivalently any) s. A geodesic in M_m is clockwise, counterclockwise, or can be extended to a geodesic through o. If γ is clockwise, then it can be mapped to a counterclockwise geodesic by an isometric involution of M_m.

Unless stated otherwise, any geodesic in M_m that we consider is either tangent to a meridian or counterclockwise. Due to this convention the Clairaut constant and the turn angle defined below are nonnegative, which will simplify notations.

2.2 Sectional Curvature

Definition 2.2.1. Let M be a Riemannian manifold of dimension 2 or higher and q a point in M. Clearly two arbitrary vectors X, Y in T_qM determine a 2-dimensional subspace $S \subset T_qM$. We define the *sectional curvature,* $G(X, Y)$, with respect to this subspace to be

$$G(X,Y) := \frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2 - g(X,Y)^2},$$

where g is the metric defined on T_qM, and Rm, called the Riemannian curvature tensor defined on M, is defined as

$$Rm(X,Y,Z,W) := \langle R(X,Y)Z, W\rangle,$$

where

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - (XY - YX)Z.$$

Convention: From this point on, *curvature* will always mean sectional curvature.

Remark 2.2.2. Below are examples that can be helpful:

1) The curvature at every point in $\mathbb{R}^n$, $n \geq 2$, is 0.

2) The curvature at every point in a metric sphere, in the induced Riemannian metric, is constant and positive.

3) Consider the hyperbola $\{x, y, z : \frac{x^2}{a^2} - \frac{z^2}{b^2} = 1;\ y = 0\}$. When we revolve this hyperbola about the z-axis, we obtain a *one-sheeted hyperboloid*, and the curvature at every point on such a surface is negative in the induced Riemannian metric. In fact, the curvature at any "saddle point" is negative.

Definition 2.2.3. Given any 2-dimensional Riemannian manifold M and the corresponding (sectional) curvature function G, we define the *total curvature* of M, $c(M)$, as

$$c(M) := \int_M G dM = \int_M G_+ dM + \int_M G_- dM,$$

provided

$$\int_M G_+ dM < \infty \quad \text{or} \quad \int_M G_- dM > -\infty,$$

where for any $q \in M$,

$$G_+(q) := \max\{0, G(q)\}, \quad G_-(q) := \min\{0, G(q)\},$$

and dM is the area element of M. If the inequalities above hold, we say that M *admits total curvature.*

Remark 2.2.4. In [CoVo], S. Cohn-Vossen proved that if M is a connected, complete, non-compact, finitely connected 2-dimensional Riemannian manifold admits a total curvature $c(M)$, then $c(M) \leq 2\pi\chi(M)$, where $\chi(M)$ is the so-called *Euler characteristic.* If M is homeomorphic to $\mathbb{R}^2$, then $\chi(M) = 1$, so $c(M) \leq 2\pi$. Hence, a rotationally symmetric plane M_m has *finite* total curvature if and only if $c(M) > -\infty$.

2.3 The Gauss-Bonnet Theorem

The Gauss-Bonnet Theorem is one of the most beautiful theorems in geometry. Below we give the version that we use for our results.

Theorem 2.3.1. *Assume M is homeomorphic to $\mathbb{R}^2$. If $P \subset M$ is a polygon with n edges each of which is an arc of a geodesic, and if $\theta_1, \theta_2, ..., \theta_n$ are the internal angles of P, then the following holds:*

$$\sum_{i=1}^{n} \theta_i = (n-2)\pi + \int_P G dM$$

If P is a triangle, the sum of the interior angles equals $\pi + \int_P GdM$. If the triangle is in $\mathbb{R}^2$ (with the standard Euclidean metric, which renders $G \equiv 0$), we recover the familiar fact that the interior angles of a triangle in a Euclidean plane add up to π.

2.4 Rotationally Symmetric Planes

We will always use M_m to denote a rotationally symmetric plane. We define a *rotationally symmetric plane* M_m as follows: For a smooth function $m\colon [0, \infty) \to [0, \infty)$ whose only zero is 0, let g_m denote the rotationally symmetric inner product on the tangent bundle to $\mathbb{R}^2$ that equals the standard Euclidean inner product at the origin and elsewhere is given in polar coordinates by $dr^2 + m(r)^2 d\theta^2$. It is well-known (see e.g. [SST03, Section 7.1]) that

- any rotationally symmetric complete smooth Riemannian metric on $\mathbb{R}^2$ is isometric to some g_m; as before M_m denotes $(\mathbb{R}^2, g_m)$;
- if $\bar{m}\colon \mathbb{R} \to \mathbb{R}$ denotes the unique odd function such that $\bar{m}|_{[0,\infty)} = m$, then g_m is a smooth Riemannian metric on $\mathbb{R}^2$ if and only if $m'(0) = 1$ and $\bar{m}$ is smooth;
- if g_m is a smooth metric on $\mathbb{R}^2$, then g_m is complete, and the sectional curvature of g_m is a smooth function on $[0, \infty)$ that equals $-\frac{m''}{m}$.

A *meridian* is a curve $\mu : [0, \infty) \to M$ emanating from the origin, o, with $\dot{\theta} \equiv 0$. A *parallel* is any locus of points on M with $r \equiv$ a constant, or equivalently, any locus of points equidistant from the origin.

Every geodesic emanating from o is a meridian; in fact, every meridian is a ray. On the other hand, a parallel is a geodesic if and only if $m'(r_0) = 0$, where r_0 is the distance from the parallel to the origin [SST03, Lemma 7.1.4].

2.5 The Cut Locus in a von Mangoldt Plane

This section culminates in Theorem 2.5.23, which has been central to our research. We start with concepts and theorems used in Theorem 2.5.23 as well as in other parts of this thesis.

Definition 2.5.1. M_m is a *von Mangoldt plane* if $G_m(r)$ is non-increasing in r. Examples of von Mangoldt planes are two-sheeted hyperboloids and paraboloids.

2.5.1 Conjugate and Focal Points

The notions of $conjugate\ points$ and $focal\ points$ are founded on understanding the effect of curvature on nearby geodesics. Our discussion below is closely modeled on expositions in [Lee], [Car].

Let $\gamma : [a, b] \to M$ be a geodesic. Then $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ is a *variation through geodesics* if each of the curves $\Gamma_s(t) = \Gamma(s, t)$ is also a geodesic. Now put $\frac{\partial \Gamma}{\partial s}(0, t) = J(t)$. It is well known that the variation field $J(t)$ satisfies the *Jacobi equation*:

$$J'' + R(J, \dot{\gamma})\dot{\gamma} = 0, \tag{2.5.2}$$

where

$$J' = \nabla_{\frac{\partial}{\partial s}} J \text{ and } J'' = \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial s}} J.$$

Any vector field J along a unit-speed geodesic γ that satisfies the above equation is called a *Jacobi field*, and every Jacobi field along a geodesic γ is the variation field of some variation of γ through geodesics.

Given a geodesic γ joining $p, q \in M$, we say that q is *conjugate to p along γ* if there is a Jacobi field along γ vanishing at p and q but not identically zero. That is, if p, q are conjugate, there exists a field of variation through geodesics not identically zero that vanishes at p and q. If q is conjugate to p along γ, then γ cannot be a minimal geodesic beyond p.

The idea of conjugate points extends to the idea of a *focal point* to a submanifold $N \subset M$. Given a geodesic $\gamma : [0, \ell] :\to M$ with $\gamma(0) = q \in N$ and $\dot{\gamma}(0) \in (T_q M)^{\perp}$, consider the geodesic variation

$$\Gamma : (-\epsilon, \epsilon) \times [0, \ell] \to M$$

such that for $s \in (-\epsilon, \epsilon)$ and $t \in [0, \ell]$, each $\Gamma_s(t)$ is a geodesic, $\Gamma_s(0) = \alpha(s) \in N$, and

$$A(s) = \frac{\partial \Gamma}{\partial t}(s, 0) \in (T_{\alpha(s)} N)^{\perp}.$$

It is well known that $J(t) = \frac{\partial \Gamma}{\partial s}(0, t)$ is a Jacobi field along γ. If $J(t)$ is not everywhere zero on $\gamma|_{[0,\ell]}$, then the point $q' = \gamma(\ell)$ is called a *focal point* of N if $J(\ell) = 0$.

2.5.2 The Sturm Comparison Theorem

Below we give a statement of the Sturm Comparison Theorem:

Theorem 2.5.3. *Let the functions $f_1(t)$ and $f_2(t)$ be continuous on $[0,\infty)$, and assume $f_1(t) \geq f_2(t)$. For each $f_i(t)$, $i = 1, 2$, let $u_i(t)$ be the solution to*

$$u_i''(t) + f_i(t)u_i(t) = 0, \tag{2.5.4}$$

where $u_i = 0$ and $u_i' = 1$ at $t = 0$. Also let a_1, a_2 be the first zeros after $t = 0$ of $u_1(t)$ and $u_2(t)$ respectively. Then we have

$$a_2 \geq a_1 \text{ and } u_2(t) \geq u_1(t) \tag{2.5.5}$$

for any $t \in [0, a_1]$.

2.5.3 The Structure of a Cut Locus in a Rotationally Symmetric Plane

We start with some preliminaries; this section culminates in Theorem 2.5.23. The discussion below is based on expositions in [SST03].

For any $q \in M$, let C_q denote the cut locus of q. For any $x \in C_q$, let $\Gamma(q,x)$ denote the set of minimal geodesics connecting q to x. Unless otherwise stated, we will assume that the cut locus of any point consists of more than one point.

Definition 2.5.6. Let ϵ be small enough so that $B_\epsilon(x)$ is a convex ball. We define a *sector* as a component of $B_\epsilon(x)\backslash\Gamma(x,q)$; If $|\Gamma(q,x)| = n < \infty$, then there exist n sectors at x. The angle at x for each sector is called an *inner angle*.

Definition 2.5.7. A *Jordan arc* is an injective continuous map from an open or closed interval of $\mathbb{R}$ into M.

Definition 2.5.8. A subset T of M is a *tree* if any two points are connected with a unique Jordan arc. A point $x \in T$ is an *endpoint* if $T \setminus x$ remains connected.

Theorem 2.5.9. (Consequence of Theorem 4.2.1, [SST03]) *If M is a complete simply connected 2-manifold, then C_q is a tree for each $q \in M$.*

Lemma 2.5.10. (Lemma 4.3.7, [SST03]) *A point x in the cut locus of any $q \in M$ is an endpoint of the cut locus if and only if x admits exactly one sector.*

Remark 2.5.11. If $x \in C_q$ admits only one sector, then x must be conjugate to q.

Definition 2.5.12. Let x be a cut point of $q \in M$. Then x is a *normal cut point* of q if there exist exactly two minimal geodesics connecting q to x and if x is not a first conjugate point of q along either of the geodesics.

Lemma 2.5.13. (Proposition 4.2.2, [SST03]) *If $x \in C_q$ is a normal cut point, then near x, C_q is a smooth curve bisecting each of the inner angles of the two sector at x.*

Lemma 2.5.14. (Lemma 7.3.2, [SST03]) *Given $q \in M_m$ with $\theta_q = 0$, assume that for $x_1, x_2 \in M_m$,*

$$r_{x_1} = r_{x_2}, \quad 0 \leq \theta_{x_1} < \theta_{x_2} \leq \pi.$$

Then $d(q, x_1) < d(q, x_2)$.

Lemma 2.5.15. (Corollary 4.2.1, [SST03]) *The set of normal cut points is open dense in C_q.*

Lemma 2.5.16. (Lemma 7.3.2, [SST03]) *Suppose M_m is von Mangoldt. If $C_q \neq \emptyset$ for any $q \in M_m$, then q is conjugate along τ_q to some point $\tau_q(t_0)$.*

Lemma 2.5.17. (Originally a part of Theorem 2.5.23, by M. Tanaka) *Suppose $q \in M_m$ and $C_q \neq \emptyset$. Then $\tau_q[t_0, \infty) \subset C_q$, where $\tau_q(t_0) > d(q, o)$ is the first conjugate point of q along τ_q.*

Proof. It is clear that $t_0 > d(q, o)$ because all meridians are rays, and the meridian emanating from o and going through q must be the unique minimal geodesic connecting the two points. By Lemma 2.5.16, there exists $t_0 \in (d(q, o), \infty)$ such that $\tau_q(t_0)$ is the first conjugate point of q along τ_q. A geodesic does not minimize beyond its first conjugate point, so for all $t > t_0$, there exists a minimal geodesic α connecting q to $\tau_q(t)$ that is distinct from τ_q. Through the involution on M_m fixing $\mu_q \cup \tau_q$, we obtain the mirror-image minimal geodesic β also connecting q to $\tau_q(t)$, implying that τ_q is the cut point of α, β. Since the above applies to all $t > t_0$, we have $\tau_q[t_0, \infty) \subset C_q$. □

Lemma 2.5.18. *Given $q \in M_m$, let $c : [0, a] \to C_q$ be a Jordan arc connecting an endpoint $c(0)$ of C_q to a point $c(a) \in C_q \cap \tau_q$. Let $c(t_n)$ be a normal cut point in $c(0, a]$, and let α, β be the two minimal geodesics connecting q to $c(t_n)$. Then the image of α, β must bound a region containing $c[0, t_n)$.*

Proof. By construction α, β bound a region R. By Lemma 2.5.13, near $c(t_n)$, C_q is a smooth curve bisecting the inner angles of the two sectors at $c(t_n)$. Hence for $\epsilon > 0$ small enough, $c(t_n - \epsilon, t_n)$ lies in R. Since α, β are distance-minimizing, they cannot intersect C_q in their interiors. Hence $c[0, t_n)$ must lie in R. □

Lemma 2.5.19. *Given $q \in M_m$, let $c : [0, a] \to C_q$ be a Jordan arc such that $c(0)$ is an endpoint of C_q, $c[0, a) \cap \tau_q = \emptyset$, and $c(a) \in \tau_q$. Let $c(t_0)$ be a cut point and $c(t_n)$ a normal cut point such that $0 < t_0 < t_n < a$. Then $\theta_{c(t_n)} > \theta_{c(t_0)}$.*

Proof. Let α, β be the two minimal geodesics connecting q to $c(t_n)$. By Lemma 2.5.18, α, β bound a region whose interior contains $c[0, t_n)$. Hence, there exists a point $t \in (0, d(q, c(t_n)))$ at which either α or β, say α, achieves $\theta_{\alpha(t)} > \theta_{c(t_0)}$. Since α cannot be tangent to a meridian, $\dot{\theta}_{\alpha(s)} > 0$ always. The claim follows. $\square$

Lemma 2.5.20. *Given $q \in M_m$, let $c : [0, a] \to C_q$ be a Jordan arc such that $c(0)$ is an endpoint of C_q, $c[0, a) \cap \tau_q = \emptyset$, and $c(a) \in C_q$. Let $c(t_0)$ be a cut point and $c(t_n)$ a normal cut point such that $0 \leq t_0 < t_n < a$. Then $d(q, c(t_n)) > d(q, c(t_0))$.*

Proof. Let α, β be the two minimal geodesics joining q to $c(t_n)$. By Lemma 2.5.19, we have $\theta_{c(t_n)} > \theta_{c(t_0)}$. Now either $r_{c(t_n)} = r_{c(t_0)}$ or $r_{c(t_n)} \neq r_{c(t_0)}$. If the former holds, then by Lemma 2.5.14 we have $d(q, c(t_n)) > d(q, c(t_0))$. If the latter holds, since α, β enclose a region whose interior contains $c(t_0)$, one of the two geodesics must have a subarc that passes through a point p in the parallel containing $c(t_0)$ such that $\theta_p > \theta_{c(t_0)}$. Lemma 2.5.14 gives us $d(q, c(t_n)) = d(q, p) + d(p, c(t_n)) > d(q, c(t_0))$. $\square$

Remark 2.5.21. Under the setting of Lemma 2.5.20, let γ be a minimal geodesic connecting q to $c(t_0)$ and let α, β be the two minimal geodesics connecting q to $c(t_n)$. Note that α, β cannot intersect γ other than at q and that by Lemma 2.5.18, α, β bound a region R whose interior contains $c[0, t_n)$. So R must also contain $\gamma(0, d(q, c(t_0))]$ in its interior. In particular, for one of α, β, say β, we have $\measuredangle(\dot{\tau}_q(0), \dot{\beta}(0)) < \measuredangle(\dot{\tau}_q(0), \dot{\gamma}(0))$.

Lemma 2.5.22. *Given $q \in M_m$, let $c : [0, a] \to C_q$ be a Jordan arc such that $c(0)$ is an endpoint of C_q, $c[0, a) \cap \tau_q = \emptyset$, and $c(a) \in C_q$. Then the distance function $d(q, c(t))$ is strictly increasing on $(0, a)$.*

Proof. Let $c(t_1)$ be any point in $c[0, a)$. It suffices for us to show that given any $t_2 > t_1$, $d(q, c(t_1)) < d(q, c(t_2))$. Since by Lemma 2.5.15 normal cut points are dense in C_q and $d(q, c(t))$ is a continuous on t, it suffices to show that for any normal cut point $c(t_2)$ with $0 \leq t_1 < t_2 < a$, $d(q, c(t_1)) < d(q, c(t_2))$. But this is true by Lemma 2.5.20. $\square$

We now present the main theorem of this section:

Theorem 2.5.23. (M. Tanaka; Theorem 7.3.1, [SST03]) *If M_m is a von Mangoldt plane, then for any point $q \in M_m$, the cut locus of q equals $\tau_q[t_0, \infty)$, where $\tau_q(t_0)$ is the first conjugate point of q along τ_q.*

Proof. By Lemma 2.5.17, we already have $\tau[t_0, \infty) \subset C_q$, so we just need to show $C_q \subset \tau[t_0, \infty)$. We first show that $C_q \subset \tau_q(d(o,q), \infty)$ through contradiction. Every meridian is a ray emanating from o, so no point of $\tau_q(0, d(q,o)] \cup \mu_q(0, \infty)$ can be a cut point. By Theorem 2.5.9, C_q is a tree, so if we assume that $C_q \not\subseteq \tau_q(d(o,p), \infty)$, there must be an endpoint x of C_q with $\theta_x < \pi$. By Remark 2.5.11, q is conjugate to x.

Let α be a minimal geodesic connecting q to x. By Lemma 2.5.22 there exists a normal cut point $y \in C_q$ such that $\theta_x < \theta_y < \pi$ and $d(q,y) > d(q,x)$. By Remark 2.5.21 there exists a minimal geodesic β connecting q to y such that $\measuredangle(\dot{\tau}_q(0), \dot{\beta}(0)) < \measuredangle(\dot{\tau}_q(0), \dot{\alpha}(0))$.

Our strategy is to show that if $s \in (0, \ell(\alpha))$, then $r_{\alpha(s)} > r_{\beta(s)}$; thus we can establish that $G_m(r_{\alpha(s)}) \leq G_m(r_{\beta(s)})$ and then apply the Sturm Comparison Theorem to derive a contradiction.

For each $s \in (0, \ell(\alpha))$, since $\theta_y > \theta_x$, there exists a unique value $t(s)$ of β giving us

$$\theta_{\alpha(s)} = \theta_{\beta(t(s))}.$$

Since α, β cannot intersect in their interiors we have $r_{\beta(t(s))} < r_{\alpha(s)}$. Hence for any given s, the set

$$S_s := \{t \in (0, \ell(\beta)) \mid r_{\beta(t)} < r_{\alpha(s)}\}$$

is nonempty. Now fix $s_0 \in (0, \ell(\alpha))$. Let (a,b) be the connected component of S_{s_0} containing $t(s_0)$. If we show that $s_0 \in (a,b)$, then we will have

$r_{\alpha(s_0)} > r_{\beta(s_0)}$. If $(0, \ell(\alpha)) \subseteq (a, b)$ then $s_0 \in (a, b)$ and there is nothing to prove, so we can assume $a > 0$ or $b < \ell(\alpha)$.

We have

$$r_{\alpha(s_0)} = r_{\beta(a)} = r_{\beta(b)}, \quad 0 \leq \theta_{\beta(a)} < \theta_{\alpha(s_0)} = \theta_{\beta(t(s_0))} < \theta_{\beta(b)} < \pi$$

so the conditions for Lemma 2.5.14 are satisfied. It follows that

$$a = d(q, \beta(a)) < s_0 = d(q, \alpha(s_0)) < d(q, \beta(b)) = b,$$

implying $s_0 \in (a, b)$ and therefore $r_{\beta(s_0)} < r_{\alpha(s_0)}$. Since s_0 was arbitrary and since M_m is von Mangoldt, this gives us $G_m(r_{\alpha(s)}) \leq G_m(r_{\beta(s)})$ for all $s \in [0, \ell(\alpha)]$. Recalling that q is conjugate to x along α and applying the Sturm Comparison Theorem, we have that q is conjugate to $\beta(t)$ along β for some $t \in (0, \ell(\alpha)]$. But this is impossible, since β minimizes the distance from q to y and $\ell(\beta) > \ell(\alpha)$. This establishes that $C_q \subset \tau_q(d(q, o), \infty)$.

It remains for us to show that $\tau_q(d(q, o), t_0) \cap C_q = \emptyset$. Proceeding by contradiction, suppose there exists $d(q, o) < t < t_0$ such that $x := \tau_q(t)$ is a cut point of q along τ_q. Since q is not conjugate to x along τ_q, there exists a geodesic γ emanating from q, different from τ_q, that also minimizes the distance to x; note that τ_q and γ bound a relatively compact domain R. There exists a geodesic σ emanating from q that lies in R for small t. Since τ_q, γ are minimizing up to x, the cut point of σ cannot be in their interior and; hence σ must intersect x. But since σ can be any geodesic that lies in R for small t, this implies that q is conjugate to x along τ_q, a contradiction.

□

Chapter 3

The Soul Theorem

Some of our main results in chapters 5 and 6 pertain to the set of souls in a rotationally symmetric plane of nonnegative sectional curvature. We therefore present below the Soul Theorem, including an outline of its proof. Our discussion is based on information in [ChEb] and [GrWal]. The Soul Theorem is as follows:

Theorem 3.0.1. (Soul Theorem) *Let M be a noncompact complete Riemannian manifold with everywhere nonnegative sectional curvature. Then there exists a compact, totally convex, boundaryless submanifold $S \subset M$, called a soul, such that M is homeomorphic to the normal bundle over S.*

Remark 3.0.2. In this chapter, it will always be assumed that M has everywhere nonnegative sectional curvature.

Remark 3.0.3. We start with some preliminaries. Theorems 3.0.6 and 3.0.10 below are special cases of the corresponding original theorems, adapted for our needs. Theorem 3.0.10 is due to Berger but is often called the Second Rauch Comparison Theorem.

Definition 3.0.4. Given any $q \in M$, we say that $r > 0$ is the *injectivity radius at q* if r is the largest value for which the exponential function maps $B_r(0) \subset T_qM$ diffeomorphically onto $B_r(q) \subset M$.

Definition 3.0.5. Consider two segments γ_1, γ_2 that meet at a point p such that $\gamma_1(\ell(\gamma_1)) = \gamma_2(0) = p$, and let $\theta := \pi - \measuredangle(\dot{\gamma}_1(\ell(\gamma_1)), \dot{\gamma}_2(0))$. We say that γ_1, γ_2 is a *hinge* and that θ is the angle formed by the hinge.

Theorem 3.0.6. (Rauch I; Theorem 3.2.1, [GrWal]) *Let $\gamma_i : [0,1] \to M$, $i = 1, 2$, be a hinge at q, and suppose $\ell(\gamma_i)$ is less than the injectivity radius at q. Then the distance between $\gamma_1(1)$ and $\gamma_2(1)$ is less than or equal to the distance between the endpoints of the comparison angle with same lengths and angle in $\mathbb{R}^2$.*

Definition 3.0.7. A subset $S \subset M$ is *totally geodesic* if every geodesic in S is also a geodesic in M.

Definition 3.0.8. We say that a vector field X along a curve γ is a *parallel vector field along* γ if $\nabla_{\dot{\gamma}} X \equiv 0$. If γ is a geodesic, $\dot{\gamma}$ is a parallel vector field along γ.

Definition 3.0.9. A subset $S \subset M$ is said to be *flat* if the curvature is everywhere zero on S.

Theorem 3.0.10. (Rauch II; Theorem 3.2.2, [GrWal]) *Let $c : [0, a] \to M$ be a geodesic, X a parallel vector field along c, and $\gamma : [0, a] \to M$ the curve given by $\gamma(t) = \exp_{c(t)} X(t)$. If for all $s \in (0, 1)$ none of the geodesics $s \mapsto \exp sX(t)$ has a focal point, then $\ell(\gamma) \leq a$. If furthermore $\ell(\gamma) = a$, then the region defined by*

$$V : [0, a] \times [0, 1] \to M, \quad (t, s) \mapsto \exp_{c(t)} sX(t)$$

is totally geodesic and flat.

(See Section 2.5.1 for discussion on focal points.)

Theorem 3.0.11. (Toponogov Comparison Theorem; Theorem 3.2.3, [GrWal]) *Let γ_i be the sides of a geodesic triangle in M, and let θ_i be the angle opposite γ_i, $i = 0, 1, 2$. Assume γ_1, γ_2 are minimal geodesics that satisfy $\ell(\gamma_1) + \ell(\gamma_2) \geq \ell(\gamma_0)$. Then there exists a triangle in $\mathbb{R}^2$ with sides $\tilde{\gamma}_i$ and angles $\tilde{\theta}_i$ such that $\ell(\gamma_i) = \ell(\tilde{\gamma}_i)$ for all i and $\theta_i \geq \tilde{\theta}_i$ for $i = 1, 2$.*

Remark 3.0.12. We now outline the proof of the Soul Theorem, which is essentially a procedure, often called *soul construction*, that can be applied to *any* noncompact manifold M of nonnegative curvature to obtain a soul.

1) Fix a point $p \in M$ and a ray γ emanating from p. Recall from Remark 2.0.9 that if M is noncompact and complete, then every point of M has at least one ray emanating from it.

Definition 3.0.13. Given a ray γ emanating from $q \in M$, we define a *horoball for* γ as

$$B_\gamma := \bigcup_{t>0} B_t(\gamma(t)),$$

where

$$B_t(\gamma(t)) := \{q \in M \mid d(\gamma(t), q) < t\}.$$

Note that $B_{t_1}(\gamma(t_1)) \subset B_{t_2}(\gamma(t_2))$ for $t_1 < t_2$.

2) Theorem 3.0.15 is the fundamental ingredient of soul construction.

Definition 3.0.14. A set $S \in M$ is *totally convex* if any geodesic in M connecting two points of S lies entirely in S.

Theorem 3.0.15. (Theorem 3.2.4, [GrWal]) *$M \setminus B_\gamma$ is a closed totally convex set.*

Proof. Since B_γ is open, $M \setminus B_\gamma$ is closed. We prove total convexity by contradiction. Suppose there exists a geodesic $\alpha : [0,1] \to M$ such that $\alpha(0), \alpha(1) \in M \setminus B_\gamma$ but $\alpha(s) \in B_\gamma$ for some $s \in (0,1)$. It follows from definitions that there exists some $t_0 > 0$ such that $\gamma(s) \in B_{t_0}(\gamma(t_0))$. Note that $t_0 > d(\gamma(t_0), \alpha(s))$; set $\epsilon := t_0 - d(\alpha(s), \gamma(t_0))$. We have for all $t \geq t_0$

$$d(\alpha(s), \gamma(t)) \leq t - \epsilon. \tag{3.0.16}$$

Fix some t such that

$$t > \max\{t_0, \ell(\alpha), \frac{\ell^2(\alpha)}{\epsilon}\}. \tag{3.0.17}$$

Let s_0 be the value at which α is closest to $\gamma(t)$. Set $\alpha_0 := \alpha|_{[0,s_0]}$, let γ_0 be a minimal geodesic joining $\alpha(0)$ to $\gamma(t)$, and let γ_{s_0} be a minimal geodesic joining $\alpha(s_0)$ to $\gamma(t)$. Note that the above geodesics define a triangle. Let θ denote the angle at $\alpha(s_0)$; note that since $\alpha(s_0)$ is the point in α closest to $\gamma(t)$ and $s \in (0,1)$, $\theta = \frac{\pi}{2}$. We will apply the Toponogov theorem to derive a contradiction regarding the measure of θ. Our triangle satisfies the inequality in the condition of the Toponogov theorem: Since $\alpha(0) \notin B_\gamma$, we have $\ell(\gamma_0) > t$, implying $\ell(\gamma_0) + \ell(\gamma_{s_0}) > t > \ell(\alpha) > \ell(\alpha_0)$. Hence we conclude that there exists a comparison triangle in $\mathbb{R}^2$ such that $\tilde{\theta} \leq \theta = \frac{\pi}{2}$.

On the other hand, (3.0.16) and (3.0.17) give us

$$\ell(\gamma_{s_0}) \leq t - \epsilon < \ell(\gamma_0) - \epsilon.$$

Now we apply the law of cosines in $\mathbb{R}^2$:

$$\cos\tilde{\theta} = \frac{\ell^2(\alpha_0) + \ell^2(\gamma_{s_0}) - \ell^2(\gamma_0)}{2\ell(\alpha_0)\ell(\gamma_{s_0})}$$

$$= \frac{\ell(\gamma_{s_0}) + \ell(\gamma_0)}{2\ell(\gamma_{s_0})} \cdot \frac{\ell(\gamma_{s_0}) - \ell(\gamma_0)}{\ell(\alpha_0)} + \frac{\ell(\alpha_0)}{2\ell(\gamma_{s_0})}$$

$$< \frac{1}{2\ell(\gamma_{s_0})}(\ell(\alpha_0) - \epsilon\frac{\ell(\gamma_{s_0}) + \ell(\gamma_0)}{\ell(\alpha_0)}) < \frac{1}{2\ell(\gamma_1)}(\ell(\gamma_0) - \frac{\epsilon t}{\ell(\gamma_0)}) < 0,$$

where the last inequality holds because by (3.0.17),

$$\frac{\ell^2(\alpha_0)}{\epsilon} < \frac{\ell^2(\alpha)}{\epsilon} < t,$$

implying

$$\ell(\alpha_0) < \epsilon\frac{t}{\ell(\alpha_0)}.$$

But this is impossible, since $\cos\tilde{\theta} < 0$ implies $\tilde{\theta} > \frac{\pi}{2}$.

□

3) We now construct a compact, totally convex set. Namely, the set

$$C_0 := \bigcap\{M \setminus B_\gamma \mid \gamma \text{ is a ray}, \gamma(0) = p\}$$

is closed, compact, and totally convex. Indeed, if C_0 is not compact, then there exists a sequence of points $q_n \in C_0$ with $d(q, q_n) \to \infty$. Let γ_n denote a minimal geodesic in C_0 joining q to q_n. Then $\{\gamma_n\}$ must subconverge to a ray γ, which is impossible by the way C_0 was constructed.

4) Next, we contract the set C_0 while preserving total convexity. For a closed totally convex set C_0 with boundary and $\alpha \geq 0$, define

$$C_0^\alpha := \{q \in C_0 \mid d(q, \partial C_0) \geq \alpha\},\ C_1 := \bigcap\{C_0^\alpha \mid C_0^\alpha \neq \emptyset\}.$$

We want to prove the following theorem:

Theorem 3.0.18. (Corollary 3.2.1, [GrWal]) *C_0^α and C_1 are totally convex, and dim $C_1 <$ dim C_0.*

Theorem 3.0.18 is implied by the following theorem:

Theorem 3.0.19. (Theorem 3.2.5, [GrWal]) *Let C be a closed totally convex set with boundary in M. Then given any geodesic $\gamma : [a, b] \to C$, the distance function*

$$f : C \to \mathbb{R}, \gamma(t) \mapsto d(\gamma(t), \partial C), t \in [a, b]$$

to the boundary is concave. Furthermore, for any geodesic γ in C, assume that $f \circ \gamma$ is equal to a constant d on some interval [a, b] and consider the parallel vector field X along γ, where $t \mapsto \exp tX(a)$ is a minimal geodesic from $\gamma(a)$ to ∂C. Then for any $s \in [a, b]$, $t \mapsto \exp tX(s)$ is a minimal geodesic of length d from $\gamma(s)$ to ∂C, and the rectangle

$$V : [a, b] \times [0, d] \to C, \quad (s, t) \mapsto \exp_{\gamma(s)} tX(s)$$

is flat and totally geodesic.

The proof of Theorem 3.0.19 is based on the following idea. Let $\gamma : [\alpha, \beta] \to C$ be a geodesic. Establish concavity of $f \circ \gamma$ by showing that on a neighborhood of any $s_0 \in (\alpha, \beta)$, $f \circ \gamma$ is bounded above by the linear function $s \mapsto (f \circ \gamma)(s_0) - (\cos \phi)(s - s_0)$, where ϕ is the angle formed by γ and the geodesic segment γ_{s_0} connecting $\gamma(s_0)$ to ∂C. Theorems 3.0.6 (Rauch I) and 3.0.10 (Rauch II) are key to this proof.

Theorem 3.0.19 implies that C_0^α is convex for all $\alpha \geq 0$ in the following way: if $\gamma[0, d]$ were a geodesic such that $\gamma(0), \gamma(d) \in C_0^\alpha$ but $\gamma(s) \notin C_0^\alpha$ for some $s \in (0, d)$, then $f \circ \gamma$ would have an absolute minimum on $(0, d)$, which is impossible for a concave function.

5) If C_1 has nonempty boundary, we can repeat the above procedure finitely many times to obtain a compact, totally convex submanifold S without boundary. S is a *soul* of M.

Remark 3.0.20. A submanifold $S' \subset M$ is a soul *only if* it is the end result of the soul construction process; the fact that M is diffeomorphic to the normal bundle over S' does not in itself make S' a soul of M.

Remark 3.0.21. Determining the set of souls of a manifold is usually nontrivial because determining the set of rays emanating from a point is generally difficult. This holds true when the M is a rotationally symmetric plane, even though we already know that each soul is isometric to a point, since every rotationally symmetric plane is diffeomorphic to $\mathbb{R}^2$.

Chapter 4

Geodesics and Rays

In this chapter, we present our observations on the behavior of geodesics, with special emphasis on rays, in rotationally symmetric plane M_m. Most of the results in sections 4.3 and 4.4 are ours, and they are crucial to identifying the souls and critical points of infinity in M_m, since determining whether a point $p \in M_m$ is in either category entails analyzing the set of rays emanating from p. Some of our observations below are also used for our results in chapter 8.

4.1 The Clairaut Relation

Below is a statement of a theorem used very often in this thesis, discovered by Alexis Clairaut:

Theorem 4.1.1. *Let γ be a geodesic in a rotationally symmetric plane M_m such that γ does not intersect the origin. Let $\kappa_{\gamma(s)} := \measuredangle(\dot\gamma(s), \partial_r)$. Then there exists a constant c such that $m(r) \sin \kappa_{\gamma(s)} = c$ for all s.*

The equality in the conclusion of the theorem above is call $Clairaut's$ $relation$. Following is an outline of the proof.

If $\gamma : I \to M_m$, $\gamma(s) = (r(s), \theta(s))$ is a geodesic that does not intersect the origin, then it satisfies the differential equations

$$\ddot{r} - mm'\dot{\theta}^2 = 0, \quad \ddot{\theta} + 2\frac{m'\dot{r}\dot{\theta}}{m} = 0,$$

where m' is the derivative with respect to r and $\dot{\theta}$, $\dot{r}$ are derivatives with respect to s (and likewise for $\ddot{\theta}, \ddot{r}$).

Note that the second geodesic equation implies

$$\frac{d}{ds}(m^2(r(s))\dot{\theta}(s) = 0, \quad (m^2(r(s))\dot{\theta}(s) = c$$

where c is some constant. This equation can be rewritten as

$$m(r)\sin\kappa_{\gamma(s)} = c.$$

Remark 4.1.2. Since $0 \leq \sin\kappa_{\gamma(s)} \leq 1$ for all s, $0 \leq c \leq m(r_\gamma(s))$ where $c = m(r_\gamma(s))$ only at points where γ is tangent to a parallel and $c = 0$ when γ is tangent to a meridian.

4.2 The Turn Angle Formula

For a geodesic $\gamma : (s_1, s_2) \to M$ that does not pass through o, we define the *turn angle* T_γ as

$$T_\gamma := \int_\gamma d\theta = \int_{s_1}^{s_2} ds = \theta_{\gamma(s_2)} - \theta_{\gamma(s_1)}.$$

From our work in deriving Clairaut's relation, we have $\dot{\theta} = \frac{c}{m^2} \geq 0$, so the integral above converges to a number in $[0, \infty]$. We wish to develop a formula for obtaining the value of T_γ given $r_{\gamma(s_1)}$ and $r_{\gamma(s_2)}$. Since γ is unit speed, we have

$$\left(\frac{dr}{ds}\right)^2 + \left(m(r)\frac{d\theta}{ds}\right)^2 = 1.$$

This gives us

$$\left(\frac{ds}{d\theta}\right)^2 \left(\frac{dr}{ds}\right)^2 = \left\{1 - \left(m(r)\frac{d\theta}{ds}\right)^2\right\} \left(\frac{ds}{d\theta}\right)^2 \Longrightarrow \left(\frac{dr}{d\theta}\right)^2 = \left(\frac{d\theta}{ds}\right)^2 - m(r).$$

Recalling $\frac{ds}{d\theta} = \frac{m^2(r(s))}{c}$ and making substitution, we have

$$\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{m^2(r(s))}{c}\right)^2 - m^2(r) \Longrightarrow \frac{d\theta}{dr} = \text{sign}\left(\frac{d\theta}{dr}\right) \frac{c}{m(r)\sqrt{m^2(r) - c^2}}.$$

Sign$\left(\frac{d\theta}{dr}\right)$ is a nonzero constant if γ is not tangent to a parallel or meridian, so putting

$$F_c := \frac{c}{m(r)\sqrt{m^2(r) - c^2}},$$

we have

$$T_\gamma = \text{sign}(\frac{d\theta}{dr}) \int_{r_{\gamma(s_1)}}^{r_{\gamma(s_2)}} F_c dr \tag{4.2.1}$$

Since $c^2 \leq m^2$, this integral is finite except possibly when some $r_i := r_{\gamma(s_i)}$ is in the set $\{m^{-1}(c), \infty\}$. The integral converges at $r_i = m^{-1}(c)$ if and only if $m'(r_i) \neq 0$. Convergence of the integral at $r_i = \infty$ implies convergence of $\int_1^\infty m^{-2} dr$, and the converse holds under the assumption $\lim_{r\to\infty} \inf m(r) > c$; this assumption is true when $G_m \geq 0$ or $G'_m \leq 0$, as follows from Lemma 4.3.5 below.

Example 4.2.2. If γ is a ray in M_m that does not pass through o, then $T_\gamma \leq \pi$; else there exists s with $|\theta_{\gamma(s)} - \theta_{\gamma(0)}| = \pi$, and by symmetry the points $\gamma(s), \gamma(0)$ are joined by two segments, so γ would not be a ray.

Example 4.2.3. If T_{γ_q} is finite, then $m'(r_q) \neq 0$ and m^{-2} is integrable on $[1, \infty)$, as follows immediately from our above discussion.

Remark 4.2.4. A geodesic is called *escaping* if its image is unbounded. In particular, rays are escaping geodesics. An example of a non-escaping geodesic is a parallel that is also a geodesic. A geodesic γ is tangent to a parallel at $\gamma(s_0)$ if and only if $\dot{r}_{\gamma(s_0)} = 0$. If $\dot{r}_{\gamma(s)}$ vanishes more than once, then γ is not escaping because it is invariant under a rotation of M_m about o [SST03, Lemma 7.1.6] and therefore not escaping. Hence, a ray is tangent to a parallel at most once.

4.3 Various lemmas and theorems

Lemma 4.3.1. *If γ_q is escaping, then $m(r) > m(r_q)$ for all $r > r_q$, and $m'(r_q) > 0$.*

Proof. Since γ_q is escaping, the image of $s \to r_{\gamma_q}(s)$ contains $[r_q, \infty)$, and q is the only point where γ_q is tangent to a parallel. The Clairaut constant of γ_q is $c = m(r_q)$, hence $m(r) > m(r_q)$ for all $r > r_q$. It follows that $m'(r_q) \geq 0$. Finally, $m'(r_q) \neq 0$ else γ_q would equal the parallel through q. □

Lemma 4.3.2. *If γ is an escaping geodesic that is tangent to the parallel P_q through q, then $\gamma \setminus \{q\}$ lies in the unbounded component of $M_m \setminus P_q$.*

Proof. By reflectional symmetry and uniqueness of geodesics, γ locally stays on the same side of parallel P_q through q, i.e. γ is the union of

γ_q and its image under the reflection fixing $\mu_q \cup \tau_q$. If γ could cross to the other side of P_q at some point $\gamma(s)$, then $|r_{\gamma(s)} - r_q|$ would attain a maximum between $\gamma(s)$ and q, and at the maximum point γ would be tangent to a parallel. Since γ is escaping, it cannot be tangent to parallels more than once, hence γ stays on the same side of P_q at all times, and since γ is escaping, it stays in the unbounded component of $M_m \setminus P_q$. □

Lemma 4.3.3. *If $\gamma : [0, \infty) \to M_m$ is a geodesic with finite turn angle, then γ is escaping.*

Proof. Note that γ is tangent to parallels in at most two points, for otherwise γ is invariant under a rotation about o, and hence its turn angle is infinite. Thus after cutting off a portion of γ we may assume that it is never tangent to a parallel, so that $r_{\gamma(s)}$ is monotone. By assumption $\theta_{\gamma(s)}$ is bounded and increasing. By Clairaut's relation $m(r_{\gamma(s)})$ is bounded below, so that $m(0) = 0$ implies that $r_{\gamma(s)}$ is bounded below. If γ were not escaping, then $r_{\gamma(s)}$ would also be bounded above, so there would exist a limit of $(r_{\gamma(s)}, \theta_{\gamma(s)})$ and hence the limit of $\gamma(s)$ as $s \to \infty$, contradicting the fact that γ has infinite length.

□

Remark 4.3.4. The lemma below presents observations on the relationship between nonnegative or nonincreasing curvature and the shape of M_m.

Lemma 4.3.5. *If m^{-2} is integrable on $[1, \infty)$, then*

(1) the function $(r \log r)^{-\frac{1}{2}} m(r)$ is unbounded;
(2) if $G_m \geq 0$, then $m' > 0$ for all r;
(3) if M_m is von Mangoldt, then $m' > 0$ for all large r;
(4) if either $G_m \geq 0$ or $G'_m \leq 0$, then $m(\infty) = \infty$.

Proof. Since m^{-2} is integrable, the function $(r \log r)^{-\frac{1}{2}} m(r)$ is unbounded, and in particular, m is unbounded. If $G_m \geq 0$ everywhere, then m' is nonincreasing with $m'(0) = 1$, and the fact that m is unbounded implies that $m' > 0$ for all r. If M_m is von Mangoldt, and $G_m(\rho_0) < 0$, then $G_m < 0$ for $r \geq \rho_0$, i.e. m' is nondecreasing on $[\rho_0, \infty)$. Since m is unbounded, there exists $\rho > \rho_0$ with $m(\rho) > m(\rho_0)$ such that $\int_{\rho_0}^{\rho} m' = m(\rho) - m(\rho_0) > 0$. Hence m' is positive somewhere on (ρ_0, ρ), and therefore on $[\rho, \infty)$. Finally, since m is an unbounded increasing function for large r, the limit $\lim_{r\to\infty} m(r) = m(\infty)$ exists and equals ∞.

□

Lemma 4.3.6. *If γ_q is escaping, then $\liminf_{r\to\infty} m(r) > m(r_q)$ if and only if there is a neighborhood U of q such that γ_u is escaping for each $u \in U$.*

Proof. First, recall that $m(r) > m(r_q)$ for $r > r_q$ and $m'(r_q) > 0$ by Lemma 4.3.1. We shall prove the contrapositive: $\liminf_{r\to\infty} m(r) = m(r_q)$ if and only if there is a sequence $u_i \to q$ such that γ_{u_i} is not escaping.

If there is a sequence $z_i \in M_m$ with $r_{z_i} \to \infty$ and $m(r_{z_i}) \to m(r_q)$, then there are points $u_i \to q$ on μ_q with $m(r_{u_i}) = m(r_{z_i})$. If γ_{u_i} is escaping, then it meets the parallel through z_i, so Clairaut's relation implies that γ_{u_i} is tangent to the parallels through u_i and z_i, which cannot happen for an escaping geodesic.

Conversely, suppose there are $u_i \to q$ such that $\gamma_i := \gamma_{u_i}$ is not escaping. Let R_i be the radius of the smallest ball about o that contains γ_i, and let P_i be its boundary parallel. Note that $R_i \to \infty$ as γ_i converges to γ_q on compact sets and γ_q is escaping, and hence $\liminf_{r\to\infty} m(r) = \lim_{r\to\infty} m(R_i)$. For each i there is a sequence $s_{i,j}$ such that the r-coordinates of $\gamma_i(s_{i,j})$ converge to R_i, which implies $\kappa_{\gamma_i(s_{i,j})} \to \frac{\pi}{2}$ as

$j \to \infty$ and i is fixed. (Note that if γ_i is tangent to P_i, then $s_{i,j}$ is independent of j, namely, $\gamma(s_{i,j})$ is the point of tangency.) By Clairaut's relation, $m(R_i) = m(r_{u_i})$, hence $\liminf_{r\to\infty} m(r) = m(r_q)$. □

Remark 4.3.7. Recall that if M_m is von Mangoldt, the cut locus of any $q \neq o$ is contained in the opposite meridian. Lemmas 4.3.8 to 4.3.11 make use of this fact in establishing rules of behavior for rays in a von Mangoldt plane.

Lemma 4.3.8. *If M_m is von Mangoldt, then a geodesic $\gamma : [0,\infty) \to M_m \setminus \{o\}$ is a ray if and only if $T_\gamma \leq \pi$.*

Proof. The "only if" direction holds even when M_m is not von Mangoldt by Example 4.2.2. Conversely, if γ is not a ray, then γ meets the cut locus of q, which is a subset of the opposite meridian $\tau_{\gamma(0)}|_{(r_{\gamma(0)},\infty)}$. Thus $T_\gamma > \pi$. □

Lemma 4.3.9. *If γ is a ray in a von Mangoldt plane, and if σ is a geodesic with $\sigma(0) = \gamma(0)$ and $\kappa_{\gamma(0)} > \kappa_{\sigma(0)}$, then σ is a ray and $T_\sigma \leq T_\gamma$.*

Proof. Set $q = \gamma(0)$. If $\kappa_{\gamma(0)} = \pi$, then $\gamma = \gamma_q$, so τ_q is a ray, which in a von Mangoldt plane implies that q is a pole [SST03, Lemma 7.3.1], so that σ is also a ray. If $\kappa_{\gamma(0)} < \pi$ and σ is not a ray, then σ is minimizing until it crosses the opposite meridian $\tau_q|_{(r_q,\infty)}$ by Theorem 2.5.23. Near q the geodesic σ lies in the region of M_m bounded by γ and μ_q, and hence before crossing the opposite meridian σ must intersect γ or μ_q, so they would not be rays. Finally, $T_\sigma \leq T_\gamma$ holds as σ lies in the sector between γ and μ_q. □

Lemma 4.3.10. *If M_m is von Mangoldt and $q \neq o$, then γ_q is a ray if and only if $q \in \mathfrak{C}_m$.*

Proof. If γ_q is a ray, then $q \in \mathfrak{C}_m$ by symmetry. If $q \in \mathfrak{C}_m$, then either q is a pole and there is a ray in every direction, or q is not a pole. If q is not a pole, τ_q is not a ray [SST03, Lemma 7.3.1], hence by the definition of $\mathfrak{C}_m$ there is a ray γ with $\kappa_{\gamma(0)} \geq \pi/2$, so γ_q is a ray by Lemma 4.3.9. □

Lemma 4.3.11. *If M_m is von Mangoldt and $q \in \mathfrak{C}_m$, then $m'(r_q) > 0$ and $m(r) > m(r_q)$ for $r > r_q$.*

Proof. Immediate from Lemmas 4.3.1 and 4.3.10.

□

Remark 4.3.12. Recall that $\hat{\kappa}(r_q)$ is the maximum of the angles formed by μ_q and rays emanating from $q \neq o$, and ξ_q is the ray for which the maximum is attained. It is immediate from definitions that $q \in \mathfrak{C}_m$ if and only if $\hat{\kappa}(r_q) \geq \frac{\pi}{2}$. Lemmas 4.3.13 to 4.3.15 focus on the behavior of ξ_q and $\hat{\kappa}(r_q)$. They were suggested by the referee for the paper on which part of this thesis is based.

Lemma 4.3.13. $\mathfrak{C}_m \neq \{o\}$ *if and only if* $\liminf_{r\to\infty} m(r) > 0$ *and* $\int_1^\infty m^{-2}$ *is finite.*

Proof. The "if" direction holds because by the main result of [Tan92a] the assumptions imply that the ball of poles has a positive radius. Conversely, if $q \in \mathfrak{C}_m \setminus \{o\}$, then ξ_q is a ray different from μ_q. By [Tan92a, Lemma 1.3, Proposition 1.7] if either $\liminf\limits_{r\to\infty} m(r) = 0$ or $\int_1^\infty m^{-2} = \infty$, then μ_q is the only ray emanating from q. □

Lemma 4.3.14. *ξ_q is the limit of the segments $[q, \tau_q(s)]$ as $s \to \infty$.*

Proof. The segments $[q, \tau_q(s)]$ subconverge to a ray σ that starts at q. Since ξ_q is a ray, it cannot cross the opposite meridian $\tau_q|_{(r_q,\infty)}$. As $[q, \tau_q(s)]$ and ξ_q are minimal, they only intersect at q, and hence the angle formed by μ_q and $[q, \tau_q(s)]$ is $\geq \hat{\kappa}(r_q)$. It follows that $\kappa_{\sigma(0)} \geq \hat{\kappa}(r_q)$, which must be an equality as $\hat{\kappa}(r_q)$ is a maximum, so $\sigma = \xi_q$. □

Lemma 4.3.15. *The function* $r \to \hat{\kappa}(r)$ *is left continuous and upper semicontinuous. In particular, the set* $\{q : \hat{\kappa}(r_q) < \alpha\}$ *is open for every* α.

Proof. If $\hat{\kappa}$ is not left continuous at r_q, then there exists $\varepsilon > 0$ and a sequence of points q_i on μ_q such that $r_{q_i} \to r_q-$ and either $\hat{\kappa}(r_{q_i}) - \hat{\kappa}(r_q) > \varepsilon$ or $\hat{\kappa}(r_q) - \hat{\kappa}(r_{q_i}) > \varepsilon$. In the former case ξ_{q_i} subconverge to a ray that makes a larger angle with μ_q than ξ_q, contradicting the maximality of $\hat{\kappa}(r_q)$. In the latter case, ξ_{q_i} intersects ξ_q for some i. Therefore, by Lemma 4.3.14 the segment $[q_i, \tau_q(s)]$ intersects $[q, \tau_q(s)]$ for large enough s at a point $z \neq \tau_q(s)$, so $\tau_q(s)$ is a cut point of z, which cannot happen for a segment. This proves that $\hat{\kappa}$ is left continuous. A similar argument shows that $\limsup_{r_{q_i} \to r_q+} \hat{\kappa}(r_{q_i}) \leq \hat{\kappa}(r_q)$, so that $\hat{\kappa}$ is upper semicontinuous, which implies that $\{q : \hat{\kappa}(r_q) < \alpha\}$ is open for every α. □

Remark 4.3.16. Lemmas 4.3.8 and 4.3.10 imply that on a von Mangoldt plane $\hat{\kappa_{r_q}} \geq \frac{\pi}{2}$ if and only if $T_{\gamma_q} \leq \pi$; the equivalence is sharpened in Theorem 4.3.29. The lemmas below are needed for Theorem 4.3.29.

Lemma 4.3.17. *If* σ *is escaping and* $0 < \kappa_{\sigma(0)} \leq \frac{\pi}{2}$, *then* $T_\sigma = \int_{r_q}^{\infty} F_c(r)dr$*; moreover, if* $\kappa_{\sigma(0)} = \frac{\pi}{2}$, *then* $c = m(r_q)$.

Proof. This formula for T_σ is immediate from 4.2.1 once it is shown that $\sigma|_{(0,\infty)}$ is not tangent to a meridian or a parallel. If $\sigma|_{(0,\infty)}$ were tangent to a meridian, $\kappa_{\sigma(0)}$ would be 0 to π, which is not the case. Since σ is escaping, by Remark 4.2.4, σ is tangent to parallels at most once. If $\kappa_{\sigma(0)} = \frac{\pi}{2}$, then σ is tangent to the parallel through $\sigma(0)$, and so $\sigma|_{(0,\infty)}$ is not tangent to a parallel. If $\kappa_{\sigma(0)} < \frac{\pi}{2}$, then σ is not tangent to a parallel, else it would be tangent to a parallel through u with $r_u > r_q$, which would imply $r_{\sigma(s)} \leq r_u$ for all s by Lemma 4.3.2, which cannot happen for an escaping geodesic. □

Remark 4.3.18. To better understand the relationship between $\hat{\kappa}(r_q)$ and T_{γ_q}, we study how T_σ depends on σ, or equivalently on $\sigma(0)$ and $\kappa_{\sigma(0)}$, when σ varies in a neighborhood of a ray γ_q.

Lemma 4.3.19. *If $G_m \geq 0$ or $G'_m \leq 0$, then the function $u \to T_{\gamma_u}$ is continuous at each point u where T_{γ_u} is finite.*

Proof. If T_{γ_u} is finite, then γ_u is escaping by Lemma 4.3.3, and hence $T_{\gamma_u} = \int_{r_u}^{\infty} F_{m(r_u)}$ by Lemma 4.3.17. We need to show that this integral depends continuously on r_u.

By Lemma 4.3.1, Lemma 4.3.5, and the discussion preceding Example 4.2.2, the assumptions on G_m and the finiteness of T_{γ_u} imply that $m(r) > m(r_u)$ for $r > r_u$, m^{-2} is integrable, $m'(r_u) > 0$, and $m(\infty) = \infty$. Hence there exists $\delta > r_u$ with $m'|_{[r_u,\delta]} > 0$ and $m(r) > m(\delta)$ for $r > \delta$; it is clear that small changes in u do not affect δ.

Write $\int_{r_u}^{\infty} F_{m(r_u)} = \int_{r_u}^{\delta} F_{m(r_u)} + \int_{\delta}^{\infty} F_{m(r_u)}$. On $[r_u, \delta]$ we can write $F_{m(r_u)} = h(r, r_u)(r - r_u)^{-\frac{1}{2}}$ for some smooth function h. Since $(r - r_u)^{-\frac{1}{2}}$ is the derivative of $2(r - r_u)^{\frac{1}{2}}$, one can integrate $F_{m(r_u)}$ by parts, which easily implies continuous dependence of $\int_{r_u}^{\delta} F_{m(r_u)}$ on r_u.

Continuous dependence of $\int_{\delta}^{\infty} F_{m(r_u)}$ on r_u follows because $F_{m(r_u)}$ is continuous in r_u and is dominated by Km^{-2}, where K is a positive constant independent of small changes in r_u. □

Remark 4.3.20. Next we focus on the case when $\sigma(0)$ is fixed, while $\kappa_{\sigma(0)}$ varies near $\frac{\pi}{2}$. To get an explicit formula for T_σ we need the following.

Lemma 4.3.21. *If M_m is von Mangoldt, and γ_q is a ray, then there exists $\epsilon > 0$ such that every geodesic $\sigma : [0, \infty) \to M_m$ with $\sigma(0) = q$ and $\kappa_{\sigma(0)} \in [\frac{\pi}{2}, \frac{\pi}{2} + \epsilon]$ is tangent to a parallel exactly once, and if u is the point where σ is tangent to a parallel, then $m' > 0$ on $[r_u, r_q]$.*

Proof. If $\kappa_{\sigma(0)} = \frac{\pi}{2}$, then $\sigma = \gamma_q$, so it is tangent to a parallel only at q, as rays are escaping. If $\kappa_{\sigma(0)} > \frac{\pi}{2}$, then σ converges to γ_q on compact subsets as $\epsilon \to 0$, so for a sufficiently small ϵ the geodesic σ crosses the parallel through q at some point $\sigma(s)$ such that $\kappa_{\sigma(s)} < \frac{\pi}{2}$. Since γ_q is a ray, rotational symmetry and Lemma 4.3.9 imply that $\sigma|_{[s,\infty)}$ is a ray, so σ is escaping. Thus σ is tangent to a parallel at a point u where $r_{\sigma(s)}$ attains a minimum and is not tangent to a parallel at any other point by Remark 4.2.4. Finally, $r_u = \lim_{\epsilon \to 0} r_q$, and since $m'(r_q) > 0$ by Lemma 4.3.11, we get $m' > 0$ on $[r_u, r_q]$ for small ϵ.

□

Remark 4.3.22. Under the assumptions of Lemma 4.3.21 the Clairaut constant c of σ equals $m(r_u) = m(r_q) \sin \kappa_{\sigma(0)}$, and the turn angle of σ is given by

$$T_\sigma = \int_{r_q}^{\infty} F_{m(r_q)}(r)dr \quad \text{if} \quad \kappa_{\sigma(0)} = \frac{\pi}{2} \quad \text{and} \tag{4.3.23}$$

$$T_\sigma = \int_{r_u}^{\infty} F_c(r)dr - \int_{r_q}^{r_u} F_c(r)dr = \int_{r_q}^{\infty} F_c(r)dr + 2\int_{r_u}^{r_q} F_c(r)dr \tag{4.3.24}$$

if $\frac{\pi}{2} < \kappa_{\sigma(0)} < \frac{\pi}{2} + \varepsilon$. These integrals converge, i.e. T_σ is finite, as follows from Example 4.2.3, and Lemmas 4.3.5, 4.3.21.

Since any geodesic σ with $\sigma(0) = q$ and $\kappa_{\sigma(0)} \in [0, \frac{\pi}{2} + \varepsilon]$ has finite turn angle, one can think of T_σ as a function of $\kappa_{\sigma(0)}$ where σ varies over geodesics with $\sigma(0) = q$ and $\kappa_{\sigma(0)} \in [0, \frac{\pi}{2} + \varepsilon]$.

Remark 4.3.25. Lemma 4.3.27 below is an elementary lemma on the continuity and differentiability of the integrals 4.3.23-4.3.24, needed for Lemma 4.3.28. We start with some preparatory comments.

Given numbers $r_q > r_0 > 0$, let m be a smooth self-map of $(0,\infty)$ such that

- $m' > 0$ on $[r_0, r_q]$,
- $m(r) > m(r_q)$ for $r > r_q$,
- m^{-2} is integrable on $(1,\infty)$,
- $\liminf_{r\to\infty} m(r) > m(r_q)$.

Example 4.3.26. Suppose $G_m \geq 0$ or $G'_m \leq 0$. If γ_q is a ray on M_m, and r_0 is sufficiently close to r_q, then m satisfies the above properties by Lemma 4.3.1, Example 4.2.3, Lemma 4.3.5.

Set $c_0 := m(r_0)$ and $c_q := m(r_q)$. Let $T = T(c)$ be the function given by the integral (4.3.23) for $c = c_q$, and by the sum of integrals (4.3.24) for $c_0 \leq c \leq c_q$, where F_c is given by (4.2.1) and $r_u := m^{-1}(c)$, where m^{-1} is the inverse of $m|_{[r_0,r_q]}$.

Lemma 4.3.27. *Under the assumptions of the previous paragraph, T is continuous on $(c_0, c_q]$, continuously differentiable on (c_0, c_q), and $T'(c)\sqrt{c_q^2 - c^2}$ converges to $-\frac{1}{m'(r_q)} < 0$ as $c \to c_q-$.*

Proof. By definition T equals $\int_{r_q}^{\infty} F_c + \int_{r_u}^{r_q} F_c$ if $c \in [c_0, c_q)$ and $T = \int_{r_q}^{\infty} F_c$ if $c = c_q$. Step 1 shows that $\int_{r_q}^{\infty} F_c$ depends continuously on $c \in [c_0, c_q]$, while Step 2 establishes continuity of T at c_q. In Steps 3–4 we prove continuous differentiability and compute the derivatives of the integrals $\int_{r_q}^{\infty} F_c$, $\int_{r_u}^{r_q} F_c$ with respect to $c \in (c_0, c_q)$. Step 5 investigates the behaviour of $T'(c)$ as $c \to c_q$.

Recall that the integral $\int_a^b H_c(r)dr$ depends continuously on c if for each $r \in (a,b)$ the map $c \to H_c(r)$ is continuous, and every c has a neighborhood U_0 in which $|H_c| \le h_0$ for some integrable function h_0. If in addition each map $c \to H_c(r)$ is C^1, and every c has a neighborhood U_1 where $|\frac{\partial H_c}{\partial c}| \le h_1$ for an integrable function h_1, then $\int_a^b H_c(r)dr$ is C^1 and differentiation under the integral sign is valid; the same conclusion holds when H_c and $\frac{\partial H_c}{\partial c}$ are continuous in the closure of $U_1 \times (a,b)$.

Step 1. The integrand F_c is smooth over (r_u, ∞), because the assumptions on m imply that $m(r) > c$ for $r > r_u$.

Since $0 < c \le c_q$ we have $F_c \le F_{c_q} = \frac{c_q}{m\sqrt{m^2 - c_q^2}}$ which is integrable on (r_q, ∞). Indeed, fix $\delta > r_q$ and note that since m^{-2} is integrable on (δ, ∞), so is F_{c_q}. To prove integrability of F_{c_q} on (r_q, δ), note that $h(r) := \frac{m(r)-m(r_q)}{r-r_q}$ is positive on $[r_q, \infty)$, as $h(r_q) = m'(r_q) > 0$ and $m(r) > m(r_q)$ for $r > r_q$. Then F_{c_q} is the product of $(r - r_q)^{-1/2}$ and a function that is smooth on $[r_q, \delta]$, and hence F_{c_q} is integrable on (r_q, δ).

Thus the integrals $\int_{r_q}^{\delta} F_c(r)dr$ and $\int_{\delta}^{\infty} F_c(r)dr$ depend continuously on $c \in (0, c_q]$, and hence so does their sum $\int_{r_q}^{\infty} F_c(r)dr$.

Step 2. As $c \to c_q$, the integral $\int_{r_u}^{r_q} F_c$ converges to zero, for if K is the maximum of $(m\,m'\sqrt{m+c})^{-1}$ over the points with $r \in [r_0, r_q]$ and $c \in [c_0, c_q]$, then

$$\int_{r_u}^{r_q} F_c \le K \int_{r_u}^{r_q} \frac{m'dr}{\sqrt{m-c}} = K \int_0^{c_q - c} \frac{dt}{\sqrt{t}}$$

which goes to zero as $c \to c_q$. Thus T is continuous at $c = c_q$.

Step 3. To find an integrable function dominating $\frac{\partial F_c}{\partial c}$ on (r_q, ∞) locally in c, note that every $c \in (c_0, c_q)$ has a neighborhood of the form $(c_0, c_q - \delta)$ with $\delta > 0$, and over this neighborhood

$$\frac{\partial F_c}{\partial c} = \frac{m}{(m^2 - c^2)^{3/2}} \le \frac{m}{(m^2 - (c_q - \delta)^2)^{3/2}},$$

where the right hand side is integrable over $[r_q, \infty)$, as m^{-2} is integrable at ∞; thus

$$\frac{d}{dc}\int_{r_q}^{\infty} F_c = \int_{r_q}^{\infty} \frac{m}{(m^2-c^2)^{3/2}} dr$$

is continuous with respect to $c \in (c_0, c_q)$. This integral diverges if $c = m(r_q)$.

Step 4. To check continuity of $\int_{r_u}^{r_q} F_c$ change variables via $t := \frac{m}{c}$ so that $r = m^{-1}(tc)$. Thus $dt = m'(r)\frac{dr}{c} = n(tc)\frac{dr}{c}$ where $n(r) := m'(m^{-1}(r))$, and

$$\int_{r_u}^{r_q} F_c(r)dr = \int_1^{c_q/c} \bar{F}_c(t)dt \qquad \text{where} \qquad \bar{F}_c(t) = \frac{1}{n(tc)\, t\sqrt{t^2-1}}.$$

Since $m' > 0$ on $[r_0, r_q]$ and $n(tc) = m'(r)$, the function $\bar{F}_c$ is smooth over $(1, \frac{c_q}{c})$. To prove continuity of $\int_1^{c_q/c} \bar{F}_c$, fix an arbitrary $(u, v) \subset (c_0, c_q)$. If $c \in (u, v)$ and $t \in (1, \frac{c_q}{c})$, then $m^{-1}(tc)$ lies in the m^{-1}-image of $(u, \frac{v}{u}c_q)$, which by taking the interval (u, v) sufficiently small can be made to lie in an arbitrarily small neighborhood of $[r_0, r_q]$, so we may assume that $m' > 0$ on that neighborhood. It follows that the maximum K of $\frac{1}{n(tc)}$ over $c \in [u, v]$ and $t \in [1, \frac{c_q}{c}]$ is finite, and $|\bar{F}_c| \le \frac{K}{t\sqrt{t^2-1}}$ for $c \in (u, v)$, i.e. $|F_c|$ is locally dominated by an integrable function that is independent of c; for the same reason the conclusion also holds for $\frac{\partial \bar{F}_c}{\partial c} = -\frac{n'(tc)}{n(tc)^2\sqrt{t^2-1}}$.

Finally, given $c_* \in (c_0, c_q)$ fix $\delta \in (1, \frac{c_q}{c_*})$, and write $\int_1^{c_q/c} \bar{F}_c = \int_1^{\delta} \bar{F}_c + \int_{\delta}^{c_q/c} \bar{F}_c$ for c varying near c_*. The first summand is C^1 at c_*, as the integrand and its derivative are dominated by the integrable function near c_*. The second summand is also C^1 at c_* as the integrand is C^1 on a neighborhood of $\{c_*\} \times [\delta, \frac{c_q}{c}]$. By the integral Leibnitz rule

$$\frac{d}{dc}\int_1^{c_q/c} \bar{F}_c = -\frac{c_q}{c^2}\,\bar{F}_c\left(\frac{c_q}{c}\right) - \int_1^{c_q/c} \frac{n'(tc)\,dt}{n(tc)^2\sqrt{t^2-1}}.$$

The first summand equals $-(m'(r_q)\sqrt{c_q^2-c^2})^{-1}$, and the second summand is bounded.

Step 5. Let us investigate the behavior of $\int_{r_q}^{\infty} \frac{m}{(m^2-c^2)^{3/2}} dr$ from Step 3 as $c \to c_q-$. Fix $\delta > r_q$ such that $m' > 0$ on $[r_0, \delta]$ and write the above integral as the sum of the integrals over (r_q, δ) and (δ, ∞). The latter one is bounded. Integrate the former integral by parts as

$$\int_{r_q}^{\delta} \frac{m\, m'}{m'\,(m^2-c^2)^{3/2}} dr = -\int_{r_q}^{\delta} \frac{1}{m'} d\left(\frac{1}{\sqrt{m^2-c^2}}\right) =$$

$$\frac{1}{m'(r_q)\sqrt{c_q^2-c^2}} - \frac{1}{m'(\delta)\sqrt{\delta^2-c^2}} - \int_{r_q}^{\delta} \frac{m''\, dr}{(m')^2\sqrt{m^2-c^2}}$$

Only the first summand is unbounded as $c \to c_q-$. The terms from Step 4 and 5 enter into T' with coefficients 2 and 1, respectively, so as $c \to c_q-$

$$T'(c)\sqrt{c_q^2-c^2} \to -\frac{1}{m'(r_q)} < 0$$

as the bounded terms multiplied by $\sqrt{c_q^2-c^2}$ disappear in the limit. $\square$

Lemma 4.3.28. *If M_m is von Mangoldt, and γ_q is a ray, then there exists $\delta > \frac{\pi}{2}$ such that the function $\kappa_{\sigma(0)} \to T_\sigma$ is continuous and strictly increasing on $[\frac{\pi}{2}, \delta]$, and continuously differentiable on $(\frac{\pi}{2}, \delta]$; moreover, the derivative of T_σ is infinite at $\frac{\pi}{2}$.*

Proof. The Clairaut constant c of σ equals $m(r_u) = m(r_q)\sin\kappa_{\sigma(0)}$, so the assertion is immediate from Lemma 4.3.27. $\square$

Theorem 4.3.29. *If M_m is von Mangoldt and $q \neq o$, then*

(1) $\hat{\kappa}(r_q) > \frac{\pi}{2}$ *if and only if* $T_{\gamma_q} < \pi$.

(2) $\hat{\kappa}(r_q) = \frac{\pi}{2}$ *if and only if* $T_{\gamma_q} = \pi$.

Proof. (1) If $\hat{\kappa}(r_q) > \frac{\pi}{2}$, then any geodesic σ with $\sigma(0) = q$ and $\kappa_{\sigma(0)} \in [\frac{\pi}{2}, \hat{\kappa}(r_q)]$ is a ray, and so has turn angle $\leq \pi$. By Lemma 4.3.28, the turn angle is increasing at $\frac{\pi}{2}$, so $T_{\gamma_q} < \pi$. Conversely if $T_{\gamma_q} < \pi$, then by

Lemma 4.3.28, the turn angle is continuous at $\frac{\pi}{2}$, so any geodesic σ with $\sigma(0) = q$ and $\kappa_{\sigma(0)}$ near $\frac{\pi}{2}$ has turn angle $< \pi$, and is therefore a ray, so $\hat{\kappa}(r_q) > \frac{\pi}{2}$.

(2) follows from (1) and the fact that $\hat{\kappa}(r_q) \geq \frac{\pi}{2}$ if and only if $T_{\gamma_q} \leq \pi$. □

Remark 4.3.30. Below are two theorems, *not* proved by us, that are used to prove our results. The statement of the first is tailored to our special situation.

Lemma 4.3.31. ([SST03, Lemma 6.1.1]) *Assume that M_m contains no line. Then, for each compact subset K of M_m, there exists a number $R(K)$ such that if $q \in M_m$ satisfies $d(q, K) > R(K)$, then no ray emanating from q passes through any point on K.*

Remark 4.3.32. Theorem 4.3.34 is the famous *Splitting Theorem*, proved by J. Cheeger and D. Gromoll in 1971. (See [Pet06] for full discussion.)

Definition 4.3.33. We define *Ricci curvature* as follows: Given a unit vector $u \in T_pM$, complete it to an orthonormal basis $\{u, e_2, ..., e_n\} \subset T_pM$. Then the Ricci curvature with respect to u equals $\sum_{i=2}^{n} G(u, e_i)$, where $G(u, e_i)$ is the sectional curvature of the 2-dimensional subspace of T_pM spanned by u, e_i. In the case of M_m, since it is a 2-dimensional surface, $G_m \geq 0$ implies that the Ricci curvature ≥ 0.

Theorem 4.3.34. (Theorem 3.8, [Pet06]) If a Riemannian manifold M contains a line and has Ricci curvature ≥ 0, then M is isometric to a product $H \times \mathbb{R}$, where H has Ricci curvature ≥ 0.

4.4 Planes of Nonnegative Curvature

A key consequence of $G_m \geq 0$ is the monotonicity of the turn angle and of $\hat{\kappa}$.

Proposition 4.4.1. *Suppose that M_m has $G_m \geq 0$. If $0 < r_u < r_v$ and γ_u has finite turn angle, then $T_{\gamma_u} \leq T_{\gamma_v}$ with equality if and only if G_m vanishes on $[r_u, \infty]$.*

Proof. The result is trivial when G is everywhere zero. Since γ_u has finite turn angle, m^{-2} is integrable, and hence m is a concave function with $m' > 0$ and $m(\infty) = \infty$ by Lemma 4.3.5.

Set $x := r_q$, so that the turn angle of γ_q is $\int_x^\infty F_{m(x)}$. As $m' > 0$, we can change variables by $t := m(r)/m(x)$ or $r = m^{-1}(tm(x))$ so that

$$\int_x^\infty F_{m(x)}(r)\, dr = \int_1^{\frac{m(\infty)}{m(x)}} \frac{dt}{l(t,x)\, t\sqrt{t^2-1}} = \int_1^\infty \frac{dt}{l(t,x)\, t\sqrt{t^2-1}}$$

where $l(t,x) := m'(r)$. Computing

$$\frac{\partial l(t,x)}{\partial x} = m''(r)\, \frac{\partial r}{\partial x} = \frac{m''(r)\, t\, m'(x)}{m'(r)} = -G(r)\, \frac{t\, m'(x)}{m'(r)} \leq 0$$

we see that $l(t,x)$ is non-increasing in x. Thus if $r_u < r_v$, then $l(t,r_u) \geq l(t,r_v)$ for all t implying $T_{\gamma_u} \leq T_{\gamma_v}$. The equality occurs precisely when $l(t,x)$ is constant on $[1,\infty)\times[r_u,r_v]$, or equivalently, when $G(m^{-1}(tm(x)))$ vanishes on $[1,\infty)\times[r_u,r_v]$, which in turn is equivalent to $G = 0$ on $[r_u,\infty)$, because $tm(x)$ takes all values in $(m(r_u),\infty)$ so $m^{-1}(tm(x))$ takes all values in (r_u,∞). $\square$

Lemma 4.4.2. *If $G_m \geq 0$, then $\hat{\kappa}$ is non-increasing in r.*

Proof. Let u_1, u_2, v be points on μ_v with $0 < r_{u_1} < r_{u_2} < r_v$. By Lemma 4.3.14 the ray ξ_{u_i} is the limit of geodesics segments that join

u_i with points $\tau_v(s)$ as $s \to \infty$. The segments $[u_1, \tau_v(s)]$, $[u_2, \tau_v(s)]$ only intersect at the endpoint $\tau_v(s)$ for if they intersect at a point z, then z is a cut point for $\tau_v(s)$, so $[\tau_v(s), u_i]$ cannot be minimizing. Hence the geodesic triangle with vertices u_1, v, $\tau_v(s)$ contains the geodesic triangle with vertices u_2, v, $\tau_v(s)$. Since $G_m \geq 0$, the former triangle has larger total curvature, which is finite as M_m has finite total curvature. As m only vanishes at 0, concavity of m implies that m is non-decreasing.

If m is unbounded, Clairaut's relation implies that the angles at $\tau_v(s)$ tend to zero as $s \to \infty$. By the Gauss-Bonnet theorem $\kappa_{\xi_1(0)} - \kappa_{\xi_2(0)}$ equals the total curvature of the "ideal" triangle with sides ξ_1, ξ_2, $[u_1, u_2]$. Thus $\hat{\kappa}(r_{u_1}) \geq \hat{\kappa}(r_{u_2})$ with equality if and only if G_m vanishes on $[r_{u_1}, \infty)$.

If m is bounded, then $\int_1^\infty m^{-2} = \infty$, so by [Tan92a, Proposition 1.7] the only ray emanating from q is μ_q so that $\hat{\kappa} = 0$ on $M_m \setminus \{o\}$. For future use note that in this case the angle formed by $\mu_q = \xi_q$ and $[q, \tau_q(s)]$ tends to zero as $s \to \infty$, so Clairaut's relation together with boundedness of m imply that the angle at $\tau_q(s)$ in the bigon with sides $[q, \tau_q(s)]$ and τ_q also tends to zero as $s \to \infty$. $\square$

Remark 4.4.3. By the above proof if $G_m \geq 0$ and m^{-2} is integrable on $[1, \infty)$, then $\hat{\kappa}(r_1) = \hat{\kappa}(r_2)$ for some $r_2 > r_1$ if and only if G_m vanishes on $[r_1, \infty)$.

Chapter 5

Critical Points of Infinity in a Rotationally Symmetric Plane

Chapter 5 presents our results on the set of critical points of infinity in a rotationally symmetric plane M_m. In the case where M_m has everywhere nonnegative sectional curvature, we show in chapter 6 that a point $p \in M$ is a critical point of infinity if and only if it is a soul, so Theorem 5.1.1 applies in the same way to the set of souls as it does to the set of critical points of infinity.

5.1 Critical Points of Infinity when Curvature is Nonnegative

Our understanding of $\mathfrak{C}_m$ is most complete when $G_m \geq 0$:

Theorem 5.1.1. *Given M_m, suppose $G_m \geq 0$. Then*
(i) C_m is a closed R_m- ball centered at o for some $R_m \in [0, \infty]$.
(ii) R_m is positive if and only if $\int_1^\infty m^{-2}$ is finite.
(iii) R_m is finite if and only if $m'(\infty) < \frac{1}{2}$.
(iv) If M_m is von Mangoldt and R_m is finite, then the equation $m'(r) =$

$\frac{1}{2}$ has a unique solution ρ_m, and the solution satisfies $\rho_m > R_m$ and $G_m(\rho_m) > 0$.
(v) If M_m is von Mangoldt and R_m is finite and positive, then R_m is the unique solution of the integral equation $\int_x^\infty \frac{m(x)dr}{m(r)\sqrt{m^2(r)-m^2(x)}} = \pi$.

Proof. (i) Since rays converge to rays, $\mathfrak{C}_m$ is closed. If any $q \neq o$ is in $\mathfrak{C}_m$, rotational symmetry implies that the parallel containing q is in $\mathfrak{C}_m$. By Lemma 4.4.2, if $q' \neq o$ lies on any parallel below q, $\hat{\kappa}(r_{q'}) \geq \hat{\kappa}(r_q)$, implying that q' must be in $\mathfrak{C}_m$. Finally, we know that $o \in \mathfrak{C}_m$.

(ii) Since m is concave and positive, it is non-decreasing, so $\liminf_{r\to\infty} m > 0$, and the claim follows from Lemma 4.3.13.

(iii) We prove the contrapositive, that $M_m = C_m$ if and only if $m'(\infty) \geq \frac{1}{2}$. The latter is equivalent to $c(M_m) \leq \pi$, since $c(M_m) = 2\pi(1 - m'(\infty))$. Note that the total curvature of a subset $Z \subset M_m$ must take on a value in $[0, 2\pi]$.

Suppose $c(M_m) \leq \pi$. Fix $q \neq o$, and consider the segments $[q, \tau_q(s)]$ that by Lemma 4.3.14 converge to ξ_q as $s \to \infty$. Consider the bigon bounded by $[q, \tau_q(s)]$ and its symmetric image under the reflection that fixes $\tau_q \cup \mu_q$. As in the proof of Lemma 4.4.2 we see that the angle at $\tau_q(s)$ goes to zero as $s \to \infty$, so the sum of angles in the bigon tends to $2(\pi - \hat{\kappa}(r_q))$. By the Gauss-Bonnet theorem, the sum of the angles of the bigon for each s equals $\int_{int(B)} G \leq c(M_m) \leq \pi$, where $\mathrm{int}(B)$ is the interior of the bigon. We conclude that $\hat{\kappa}(r_q) \geq \frac{\pi}{2}$, so $q \in C_m$.

Conversely, suppose that $\mathfrak{C}_m = M_m$. Given $\epsilon > 0$, find a compact rotationally symmetric subset $K \subset M_m$ with $c(K) > c(M_m) - \epsilon$. Fix $q \neq o$ and consider the rays $\xi_{\mu_q(s)}$ as $s \to \infty$. If all these rays intersect K, then they subconverge to a line by Lemma 4.3.31, so by Theorem 4.3.34, M_m is the standard $\mathbb{R}^2$ (with the Euclidean metric $dx^2 + dy^2$), and $c(M_m) = 0 < \pi$. Thus we can assume that there exists a point v on the ray μ_q such that ξ_v is disjoint from K. Therefore, if s is large enough, then K lies inside the bigon bounded by $[v, \tau_v(s)]$ and its symmetric image under the reflection that fixes $\tau_q \cup \mu_q$. The sum of the angles in the bigon tends to $2(\pi - \hat{\kappa}(r_v))$, and by the Gauss-Bonnet theorem it is bounded below by $c(K)$. Since $v \in \mathfrak{C}_m$, we have $\hat{\kappa}(r_v) \geq \frac{\pi}{2}$, and hence $c(K) \leq \pi$. Thus $c(M_m) < \pi + \epsilon$, and since ϵ is arbitrary, we get $c(M_m) \leq \pi$, which completes the proof of (iii).

(iv) Since R_m is finite, $m'(\infty) < \frac{1}{2}$ by part (iii). As $m'(0) = 1$, the equation $m'(x) = \frac{1}{2}$ has a solution ρ_m. As $G_m \geq 0$, the function m' is nonincreasing, so uniqueness of the solution is equivalent to positivity of $G_m(\rho_m)$. Since M_m is von Mangoldt, $G_m(\rho_m) > 0$, for otherwise G_m would have to vanish for $r \geq \rho_m$, implying $m'(\infty) = m'(\rho_m) = \frac{1}{2}$, and R_m would be infinite, a contradiction.

Now we show that $\rho_m > R_m$. This is clear if $R_m = 0$ because $\rho_m \geq 0$ and $m'(0) = 1 \neq \frac{1}{2} = m'(\rho_m)$. In the case where $R_m > 0$, we prove our claim by showing that $T_{\gamma_v} > \pi$ for any $v \in M_m$ with $r_v \geq \rho_m$, for by Lemma 4.3.10, since M_m is von Mangoldt, this would imply that $v \notin \mathfrak{C}_m$. Recall that if $R_m > 0$, then m^{-2} is integrable by Lemma 4.3.13, so $m' > 0$ everywhere by the proof of Lemma 4.3.5. Hence for any $r_v \geq \rho_m$, we have $m(r_v) \geq m(\rho_m)$, which implies $tm(r_v) > m(\rho_m)$ for all $t > 1$. Thus $m^{-1}(tm(r_v)) > m^{-1}(m(\rho_m)) = \rho_m$. Applying m' to the inequality, we get in notations of Proposition 4.4.1 that $l(t, r_v) < m'(\rho_m) = \frac{1}{2}$, where the inequality is strict because $G_m(r_m) > 0$ by part (iv). Now 6.0.3 below implies

$$T_{\gamma_v} = \int_1^\infty \frac{dt}{l(t, r_v)t\sqrt{t^2 - 1}} > \int_1^\infty \frac{2dt}{t\sqrt{t^2 - 1}} = \pi.$$

(v) Since R_m is positive and finite, and M_m is von Mangoldt, there are geodesics tangent to parallels whose turn angles are $\le \pi$, and $> \pi$, respectively. By Proposition 4.4.1 the turn angle is monotone with respect to r, so let r_q be the (finite) supremum of all x such that $\int_x^\infty F_{m(x)} < \pi$. Since $\mathfrak{C}_m$ is closed, $q \in \mathfrak{C}_m$ so that $T_{\gamma_q} \le \pi$. In fact, $T_{\gamma_q} = \pi$, for if $T_{\gamma_q} < \pi$, then r_q is not maximal because by Theorems 5.2.2 and 4.3.29 the set of points q with $T_{\gamma_q} < \pi$ is open in M_m. If $G_m(r_q) > 0$, then by monotonicity r_q is a unique solution of $T_{\gamma_q} = \pi$. If $G_m(r_q) = 0$, then $G_m|_{[r_q,\infty)} = 0$ as M_m is von Mangoldt, so 6.0.3 implies that the turn angle of each γ_v with $r_v \ge r_q$ equals $\frac{\pi}{2m'(r_q)}$. So $m'(r_q) = \frac{1}{2}$ but this case cannot happen as R_m is infinite by (iii). □

Example 5.1.2. Let M_m be a paraboloid in $\mathbb{R}^3$. Since $m'(\infty) = 0$, i.e. $c(M_m) = 2\pi$, we have $\mathfrak{C}_m = \{o\}$.

5.2 Critical Points of Infinity and Poles

Theorem 5.1.1 should be compared with the following results of Tanaka:

- the set of poles in any M_m is a closed metric ball centered at o of some radius R_p in $[0, \infty]$ [Tan92b, Lemma 1.1].
- $R_p > 0$ if and only if $\int_1^\infty m^{-2}$ is finite and $\liminf_{r\to\infty} m(r) > 0$ [Tan92a].
- if M_m is von Mangoldt, then R_p is a unique solution of an explicit integral equation [Tan92a, Theorem 2.1].

It is natural to wonder when the set of poles equals $\mathfrak{C}_m$, and we answer the question when M_m is von Mangoldt:

Theorem 5.2.1. *If M_m is a von Mangoldt plane, then*

(a) *If R_p is finite and positive, then the set of poles is a proper subset of the component of $\mathfrak{C}_m$ that contains o.*

(b) $R_p = 0$ *if and only if* $\mathfrak{C}_m = \{o\}$.

In preparing for the proof of Theorem 5.2.1 we prove Theorem 5.2.2. First, we say that a ray γ in M_m *points away from infinity* if γ and the segment $[\gamma(0), o]$ make an angle $< \frac{\pi}{2}$ at $\gamma(0)$. Define $A_m \subset M_m - \{o\}$ as follows: $q \in A_m$ if and only if there is a ray that starts at q and points away from infinity; by symmetry, $A_m \subset \mathfrak{C}_m$.

Theorem 5.2.2. *If M_m is a von Mangoldt plane, then A_m is open in M_m.*

Proof. By Theorem 4.3.29 we know that $q \in A_m$ if and only if $T_{\gamma_q} < \pi$, and by Lemma 4.3.19 the map $u \to T_{\gamma_u}$ is continuous at q, so the set $\{u \in M_m \,|\, T_{\gamma_u} < \pi\}$ is open, and hence so is A_m. □

Another proof. Fix $q \in A_m$ so that $T_{\gamma_q} < \pi$ by Theorem 4.3.29. Fix $\varepsilon > 0$ such that $\varepsilon + T_{\gamma_q} < \pi$. Let P_q be the parallel through q. Then there is a ray γ with $\gamma(0) = q$ and $\kappa_{\gamma(0)} > \frac{\pi}{2}$ such that γ intersects P_q at points q, $\gamma(t)$, and the turn angle of $\gamma|_{(0,t)}$ is $< \varepsilon$.

For an arbitrary sequence $q_i \to q$ we need to show that $q_i \in A_m$ for all large i. Let $\gamma_i \colon [0, \infty) \to M_m$ be the geodesic with $\gamma_i(0) = q_i$ and $\kappa_{\gamma_i(0)} = \kappa_{\gamma(0)}$. Since γ_i converge to γ on compact sets, for large i there are $t_i > 0$ such that $\gamma_i(t_i) \in P_q$ and $t_i \to t$. The angle formed by γ and $\mu_{\gamma(t)}$ is $< \frac{\pi}{2}$. Rotational symmetry and Lemma 4.3.9 imply that if i is large, then $\gamma_i|_{[t_i,\infty)}$ is a ray whose turn angle is $\le T_{\gamma_q}$. The turn angles of $\gamma_i|_{(0,t_i)}$ converge to the turn angle of $\gamma|_{(0,t)}$, which is $< \varepsilon$. Thus $T_{\gamma_i} < T_{\gamma_q} + \varepsilon < \pi$ for large i, so that γ_i is a ray by Lemma 4.3.8, and hence $q_i \in A_m$. □

Proof of Theorem 5.2.1. (a) Let P_m denote the set of poles; it is a closed metric ball [Tan92b, Lemma 1.1]. Moreover, P_m clearly lies in the connected component A_m^o of $A_m \cup \{o\}$ that contains o, and hence in the component of $\mathfrak{C}_m$ that contains o. By Theorem 5.2.2 A_m is open in M_m, so $A_m \cup \{o\}$ is locally path-connected, and hence A_m^o is open in M_m. If P_m were equal to A_m^o, the latter would be closed, implying $A_m^o = M_m$, which is impossible as the ball has finite radius.

(b) The "if" direction is trivial as $P_m \subset \mathfrak{C}_m$. Conversely, if $\mathfrak{C}_m \neq \{o\}$, then by Lemma 4.3.13 m^{-2} is integrable and $\liminf_{r\to\infty} m(r) > 0$, so $R_p > 0$ [Tan92a]. □

Remark 5.2.3. Of course $R_p = \infty$ implies $\mathfrak{C}_m = M_m$, but the converse is not true: Theorem 7.2.1 ensures the existence of a von Mangoldt plane with $m'(\infty) = \frac{1}{2}$ and $G_m \geq 0$, and for this plane $\mathfrak{C}_m = M_m$ by Theorem 5.1.1, while R_p is finite by Remark 6.0.5.

5.3 Critical Points of Infinity in a von Mangoldt Plane with Negative Curvature

Recall that by definition, if M_m is von Mangoldt, then $G' \leq 0$. Hence, if $G(r) < 0$ at some r_0, then $G < 0$ on $[r_0, \infty)$. The theorem below collects most of what we know about $\mathfrak{C}_m$ in this case.

Theorem 5.3.1. *If M_m is a von Mangoldt plane with a point where $G_m < 0$ and such that $\liminf_{r\to\infty} m(r) > 0$, then*

(1) *M_m contains a line and has total curvature $-\infty$;*

(2) *if m' has a zero, then neither A_m nor $\mathfrak{C}_m$ is connected;*

(3) *$M_m - A_m$ is a bounded subset of M_m;*

(4) *the ball of poles of M_m has positive radius.*

Proof. By assumption there is a point of negative curvature, and since the curvature is non-increasing, outside a compact subset the curvature is bounded above by a negative constant. As $\liminf_{r\to\infty} m(r) > 0$, m is bounded below by a positive constant outside any neighborhood of 0, so $\int_0^\infty m = \infty$. Hence the total curvature $2\pi\int_0^\infty G_m(r)\, m(r)\, dr$ is $-\infty$.

Hence there exists a metric ball B of finite positive radius centered at o such that the total curvature of B is negative, and such that no point with $G \geq 0$ lies outside B. By [SST03, Theorem 6.1.1, page 190], for any $q \in M_m$ the total curvature of the set obtained from M_m by removing all rays that start at q is in $[0, 2\pi]$. So for any q there is a ray that starts at q and intersects B.

If q is not in B, then the ray points away from infinity, so $q \in A_m$ and any point on this ray is in $\mathfrak{C}_m$. Thus $M_m - A_m$ lies in B. Since $\mathfrak{C}_m \neq \{o\}$, Theorem 5.2.1 implies that $R_p > 0$. Letting q run to infinity, the rays subconverge to a line that intersects B (see e.g. [SST03, Lemma 6.1.1, page 187].

If $m'(r_q) = 0$, then the parallel through q is a geodesic but not a ray, so Lemma 4.3.10 implies that no point on the parallel through q is in $\mathfrak{C}_m$. Since $\mathfrak{C}_m$ contains o and all points outside a compact set, $\mathfrak{C}_m$ is not connected; the same argument proves that A_m is not connected. □

Example 5.3.2. Here we modify [Tan92b, Example 4] to construct a von Mangoldt plane M_m such that m' has a zero and neither A_m nor $\mathfrak{C}_m$ is connected. Given $a \in (\frac{\pi}{2}, \pi)$ let $m_0(r) = \sin r$ for $r \in [0, a]$, and define m_0 for $r \geq a$ so that m_0 is smooth, positive, and $\liminf_{r\to\infty} m_0 > 0$. Thus $K_0 := -\frac{m_0''}{m_0}$ equals 1 on $[0, a]$. Let K be any smooth nonincreasing function with $K \leq K_0$ and $K|_{[0,a]} = 1$. Let m be the solution of 7.1.7;

note that $m(r) = \sin(r)$ for $r \in [0, a]$ so that m' vanishes at $\frac{\pi}{2}$. By Sturm comparison $m \geq m_0 > 0$, and hence M_m is a von Mangoldt plane. Since $m'(a) < 0$ and $m > 0$ for all $r > 0$, the function m cannot be concave, so $K = G_m$ eventually becomes negative, and Theorem 5.3.1 implies that A_m and $\mathfrak{C}_m$ are not connected.

Example 5.3.3. Here we construct a von Mangoldt plane such that $m' > 0$ everywhere but A_m and $\mathfrak{C}_m$ are not connected. Let M_n be a von Mangoldt plane such that $G_n \geq 0$ and $n' > 0$ everywhere, and R_n is finite (where R_n is the radius of the ball $\mathfrak{C}_n$). This happens e.g. for any paraboloid, any two-sheeted hyperboloid with $n'(\infty) < \frac{1}{2}$, or any plane constructed in Theorem 7.2.1 with $n'(\infty) < \frac{1}{2}$. Fix $q \notin \mathfrak{C}_n$. Then γ_q has turn angle $> \pi$, so there is $R > r_q$ such that $\int_{r_q}^{R} F_{n(r_q)} > \pi$. Let G be any smooth non-increasing function such that $G = G_n$ on $[0, R]$ and $G(z) < 0$ for some $z > R$. Let m be the solution of (7.1.7) with $K = G$. By Sturm comparison $m \geq n > 0$ and $m' \geq n' > 0$ everywhere; see Remark 7.1.10. Since $m = n$ on $[0, R]$, on this interval we have $F_{m(r_q)} = F_{n(r_q)}$, so in the von Mangoldt plane M_m the geodesic γ_q has turn angle $> \pi$, which implies that no point on the parallel through q is in $\mathfrak{C}_m$. Now parts (3)-(4) of Theorem 5.3.1 imply that A_m and $\mathfrak{C}_m$ are not connected.

5.4 Creating Annuli Free of Critical Points of Infinity

Remark 5.4.1. It is natural for one to be interested in subintervals of $(0, \infty)$ that are disjoint from $r(\mathfrak{C}_m)$, as e.g. happens for any interval on which $m' \leq 0$, or for the interval (R_m, ∞) in Theorem 5.1.1. To this end we prove Theorem 5.4.3. Theorem 5.4.2 is needed for us to prove Theorem 5.4.3.

Theorem 5.4.2. *Let M_m be a von Mangoldt plane such that $m'|_{[0,y]} > 0$ and $m'|_{[x,y]} < \frac{1}{2}$. Set $f_{m,x}(y) := m^{-1}(\cos(\pi b)\, m(y))$, where b is the maximum of m' on $[x,y]$. If $x \le f_{m,x}(y)$, then $r(\mathfrak{C}_m)$ and $[x, f_{m,x}(y)]$ are disjoint.*

Proof. Set $f := f_{m,x}$. Arguing by contradiction assume there exists $q \in \mathfrak{C}_m$ with $r_q \in [x, f(y)]$. Then γ_q has turn angle $\le \pi$, so if $c := m(r_q)$, then

$$\pi \ge \int_{r_q}^{\infty} \frac{c\,dr}{m\sqrt{m^2-c^2}} > \int_{r_q}^{y} \frac{c\,dr}{m\sqrt{m^2-c^2}} = \int_{c}^{m(y)} \frac{c\,dm}{m'(r)\,m\sqrt{m^2-c^2}} \ge$$

$$\int_{c}^{m(y)} \frac{c\,dm}{b\,m\sqrt{m^2-c^2}} = \frac{1}{b}\arccos\left(\frac{c}{m(y)}\right)$$

so that $\pi b > \arccos\left(\frac{c}{m(y)}\right)$, which is equivalent to $\cos(\pi b)\, m(y) < m(r_q)$.

On the other hand, $m(f(y))$ is in the interval $[0, m(y)]$ on which m^{-1} is increasing, so $f(y) < y$, and therefore m is increasing on $[x, f(y)]$. Hence $r_q < f(y)$ implies $m(r_q) < m(f(y)) = \cos(\pi b)\, m(y)$, which is a contradiction. □

Theorem 5.4.3. *Let M_n be a von Mangoldt plane with $G_n \ge 0$, $n(\infty) = \infty$, and such that $n'(x) < \frac{1}{2}$ for some x. Then for any $z > x$ there exists $y > z$ such that if M_m is a von Mangoldt plane with $n = m$ on $[0,y]$, then $r(\mathfrak{C}_m)$ and $[x,z]$ are disjoint.*

Proof. We use the notation in Theorem 5.4.2. The assumptions on n imply $n' > 0$, $n'|_{[x,\infty)} < \frac{1}{2}$, and $b = n'(x)$. Hence $f_{n,x}(\infty) = \infty$. In particular, if y is large enough, then $f_{n,x}(y) > z > x$; fix y that satisfies the inequality. Now if M_m is any von Mangoldt plane with $m = n$ on $[0,y]$, then $f_{m,x}(y) = f_{n,x}(y)$, so M_m satisfies the assumptions of Theorem 5.4.2, so $[x,z]$ and $r(\mathfrak{C}_m)$ are disjoint.

□

Remark 5.4.4. In general, if M_m, M_n are von Mangoldt planes with $n = m$ on $[0, y]$, then the sets $\mathfrak{C}_m$, $\mathfrak{C}_n$ could be quite different. For instance, if M_n is a paraboloid, then $\mathfrak{C}_n = \{o\}$, but by Example 5.3.3 for any $y > 0$ there is a von Mangoldt M_m with some negative curvature such that $m = n$ on $[0, y]$, and by Theorem 5.3.1 the set $M_m - \mathfrak{C}_m$ is bounded and $\mathfrak{C}_m$ contains the ball of poles of positive radius.

Chapter 6

Souls in a Rotationally Symmetric Plane

Recall that the soul construction takes as input a basepoint in an open complete manifold N of nonnegative sectional curvature and produces a compact totally convex submanifold S without boundary, called a *soul*, such that N is diffeomorphic to the normal bundle to S. Thus if N is contractible, as happens for M_m, then S is a point. The soul construction also gives a continuous family of compact totally convex subsets that starts with S and ends with N, and according to [Men97, Proposition 3.7] $q \in N$ is a critical point of infinity if and only if there is a soul construction such that the associated continuous family of totally convex sets drops in dimension at q. In particular, any point of S is a critical point of infinity, which can also be seen directly; see the proof of [Mae75, Lemma 1]. In Theorem 6.0.1 we prove conversely that every point of $\mathfrak{C}_m$ is a soul; for this M_m need not be von Mangoldt.

Theorem 6.0.1. *If M_m is a plane of nonnegative curvature, then the set of souls is equal to the set of critical points of infinity.*

As we shall see below, in the case of M_m with $G \geq 0$, the soul construction with basepoint $q \in \mathfrak{C}_m \setminus \{o\}$ takes no more than two steps; more precisely, deleting the horoballs for rays emanating from q results either in $\{q\}$ or in a segment with q as an endpoint. In the latter case the soul is the midpoint of the segment. In what follows, we let B_σ denote the (open) horoball for a ray σ with $\sigma(0) = q$, i.e. the union over $t \in [0, \infty)$ of the metric balls of radius t centered at $\sigma(t)$. Let H_σ denote the complement of B_σ in the ambient Riemannian manifold.

We start with a lemma:

Lemma 6.0.2. *Let σ be a ray in a complete Riemannian manifold M, and let $q = \sigma(0)$. Then for any nonzero $v \in T_qM$ that makes an acute angle with σ, the point $\exp_q(tv)$ lies in the horoball B_σ for all small $t > 0$.*

Proof of Theorem 6.0.1. This follows from the definition of a horoball, for if Υ denotes the image of $t \to \exp_q(tv)$, then $\lim_{s \to +0} \frac{d(\sigma(s), \Upsilon}{d(\sigma(s), q)} = \sin \measuredangle(v'(0), \sigma'(0)) < 1$, so B_σ contains a subsegment of $\Upsilon - \{q\}$ that approaches q. $\square$

For $q \in \mathfrak{C}_m$, let C_q denote the complement in M_m of the union of the horoballs for rays that start at q; note that C_q is compact and totally convex. If C_q equals $\{q\}$, then q is a soul. Otherwise, C_q has positive dimension and $q \in \partial C_q$. Set $\gamma := \xi_q$; thus γ is a ray.

Case 1. Suppose $\pi/2 < \hat{k}(r_q) < \pi$. Let $\bar{\gamma}$ be the clockwise ray that is mapped to γ by the isometry fixing the meridian through q. travels in the clockwise direction.) We next show that q is the intersection of the complements of the horoballs for rays μ_q, γ, $\bar{\gamma}$, implying that q is a soul for the soul construction that starts at q. As $\hat{k}(r_q) > \pi/2$, any nonzero $v \in T_qM_m$ forms angle $< \pi/2$ with one of $\mu'(0)$, $\gamma'(0)$, $\bar{\gamma}'(0)$. So $\exp_q(tv)$ must lie in one of the three horoballs above and hence $\exp_q(tv)$ cannot lie in the intersection of H_{μ_q}, H_γ, $H_{\bar{\gamma}}$ for small t. Since the intersection is totally convex, it is $\{q\}$. (Recall that a subset $C \subset M$ is totally convex if any geodesic of M connecting two points in C lies entirely in C. Hence if we cannot have a nontrivial geodesic emanating from q and staying inside C_q, C_q must be q; that is, q must be a soul.)

Case 2. Suppose $\hat{k}(r_q) = \frac{\pi}{2}$, so that $\gamma = \gamma_q$, and suppose G_m does not vanish along γ. By symmetry and Lemma 6.0.2 it suffices to show that every point of the segment $[o, q)$ near q lies in B_γ. Let α be the ray from o passing through q. The geodesic γ is orthogonal to α, and it suffices to show that there is a focal point w of α along γ (for this would imply that there is a family of curves near γ along which the distance from α to any point u on γ beyond w is shorter than the distance to u along γ). [Sak96], Lemma III.2.11).

Any α-Jacobi field along γ is of the form jn where n is a parallel nonzero normal vector field along γ and j solves $j''(t) + G(r_{\gamma(t)})j(t) = 0$, $j(0) = 1$, $j'(0) = 0$. Since $G \geq 0$, the function j is concave, so due to its initial values, j must vanish unless it is constant. The point where j vanishes is focal. If j is constant, then $G = 0$ along γ, which is ruled out by assumption.

Case 3. Suppose $\hat{k}(r_q) = \pi$, so that $\gamma = \tau_q$. For any vector $v \in T_qM_m$ pointing inside C_q, for small t the point $\exp_q(tv)$ is not in the horoballs for μ_q and τ_q. Hence v is tangent to a parallel, and C_q must be a subsegment of the geodesic α tangent to the parallel through q. As C_q lies outside the horoballs for μ_q and τ_q, along these rays there cannot be focal points of α, implying that G_m vanishes along μ_q and τ_q, and hence everywhere, by rotational symmetry, so that M_m is the standard $\mathbb{R}^2$, and q is a soul (recalling that every point of $\mathbb{R}^2$ is a soul).

Case 4. Suppose $\hat{\kappa}(r_q) = \frac{\pi}{2}$, so that $\gamma = \gamma_q$, and suppose that G_m vanishes along γ. We show that q is a soul by showing that *every* point in M_m is a soul. Our strategy is twofold: First we show that o must be in the horoball of γ_q. Using this fact, we then show that if we choose basepoint q appropriately, any point in M_m can be rendered a soul.

By rotational symmetry $G_m = 0$ for $r \geq r_q$, so $m(r) = ar + m(0)$ for $r \geq r_q$ where $a > 0$, as m only vanishes at 0. The turn angle of γ can be computed explicitly as

$$\int_x^\infty \frac{dr}{m(r)\sqrt{\frac{m(r)^2}{m(x)^2} - 1}} = \int_1^\infty \frac{dt}{a\,t\sqrt{t^2-1}} = -\frac{1}{a}\operatorname{arccot}(\sqrt{t^2-1})\Big|_1^\infty = \frac{\pi}{2a} \tag{6.0.3}$$

where $x := r_q$. Since γ is a ray, we deduce that $a \geq \frac{1}{2}$, for if $a < \frac{1}{2}$, then the turn angle of γ would be greater than π, implying that γ intersects τ_q.

Let $z \leq x$ be the smallest number such that $m'|_{[z,\infty)} = a$; thus there is no neighborhood of z in $(0, \infty)$ on which G_m is identically zero.

Note that $m(r) = a(r-z)+m(z)$ for $r \geq z$, so the surface $M_m - B(o, z)$ is isometric to $C - B(\bar{o}, \frac{m(r_q)}{a})$ where C is the cone with apex $\bar{o}$ such that cutting C along the meridian from $\bar{o}$ gives a sector in $\mathbb{R}^2$ of angle $2\pi a$ with the portion inside the radius $\frac{m(r_q)}{a}$ removed.

Since γ_q is a ray, Lemma 6.0.2 implies the existence of a neighborhood U_q of q such that each point in $U_q \setminus [o, q]$ lies in a horoball for a ray from q.

We now check that o lies in the horoball of γ_q. Concavity of m implies that the graph of m lies below its tangent line at z, so evaluating the tangent line at $r = 0$ and using $m(0) = 0$ gives $\frac{m(z)}{a} > z$. The Pythagorean theorem in the sector in $\mathbb{R}^2$ of angle $2\pi a$ implies that

$$d_{M_m}(\gamma_q(s), o) = \sqrt{s^2 + (x - z + \frac{m(z)}{a})^2} + z - \frac{m(z)}{a}$$

which is $< s$ for large s, implying that o is in the horoball of γ_q.

In the second phase of our proof, we show that every point of M_m is a soul. To realize q as a soul, we need to look at the soul construction with arbitrary basepoint v, which starts by considering the complement in M_m of the union of horoballs for all rays from v, which by the above is either v or a segment $[u,v]$ contained in $(o,v]$, where u is uniquely determined by v. It will be convenient to allow for degenerate segments for which $u=v$; with this convention, the soul is the midpoint of $[u,v]$. Since z is the smallest such that $G_m|_{[z,\infty)}=0$, the focal point argument of Case 2 shows that $u=v$ when $0<r_v<z$. Set $y:=r_v$, and let $e(y):=r_u$; note that $0<e(y)\le y$, and the midpoint of $[u,v]$ has r-coordinate $h(y):=\frac{y+e(y)}{2}$.

To realize each point of M_m as a soul, it suffices to show that each positive number is in the image of h. Since h approaches zero as $y\to 0$ and approaches infinity as $y\to\infty$, it is enough to show that h is continuous and then apply the Intermediate Value theorem.

Since $e(y)=y$ when $0<y<z$, we only need to verify continuity of e when $y\ge z$. Let v_i be an arbitrary sequence of points on α converging to v, where as before α is the ray from o passing through q. Set $v_i:=r_{v_i}$. Arguing by contradiction suppose that $e(y_i)$ does not converge to $e(y)$. Since $0<e(y_i)\le y_i$ and $y_i\to y$, we may pass to a subsequence such that $e(y_i)\to e_\infty\in[0,y]$. Pick any w such that r_w lies between e_∞ and $e(y)$. Thus there exists i_0 such that either $e(y_i)<r_w<e(y)$ for all $i>i_0$, or $e(y)<r_w<e(y_i)$ for all $i>i_0$. As $y\ge z$, we know that G_m vanishes along γ_v, so every α-Jacobi field along γ_v is constant. Therefore, the rays γ_{v_i} converge uniformly to γ_v as $v_i\to v$, and hence their Busemann functions b_i, b converge pointwise. Thus $b_i(w)\to b(w)$, but we had chosen w so that $b(w)$, $b_i(w)$ are all nonzero, and $\operatorname{sign}(b(w))=-\operatorname{sign}(b_i(w))$, which gives a contradiction.

Remark 6.0.4. In Cases 1, 2, and 3 the soul construction terminates in one step; namely, if $q \in \mathfrak{C}_m$, then $\{q\}$ is the result of removing the horoballs for all rays that start at q. We do not know whether the same is true in Case 4 because the basepoint v needed to produce the soul q is found implicitly via the Intermediate Value theorem, and it is unclear how v depends on q and whether $v = q$.

Remark 6.0.5. Let M_m be as in Case 4 with $m'|_{[z,\infty)} = \frac{1}{2}$. If M_m is von Mangoldt, then no point q with $r_q \geq z$ is a pole because by 6.0.3 the turn angle of γ_q is π, which by Theorem 4.3.29 cannot happen for a pole.

Chapter 7

More on von Mangoldt Planes

In this chapter, we start with gathering some facts and observations on von Mangoldt planes; the chapter culminates in Theorem 7.2.1, in which we show that we can construct a von Mangoldt plane M_m that is a cone near infinity and for which we can prescribe $m'(r)$ to take on any value in $(0, 1]$.

7.1 Some Observations

It is often useful to visualize M_m as a surface of revolution in $\mathbb{R}^3$, so we recall the following lemma (note that M_m is not assume to be von Mangoldt):

Lemma 7.1.1.
(1) *M_m is isometric to a surface of revolution in $\mathbb{R}^3$ if and only if $|m'| \leq 1$.*
(2) *M_m is isometric to a surface of revolution $(r\cos\phi,\, r\sin\phi,\, g(r))$ in $\mathbb{R}^3$ if and only if $0 < m' \leq 1$.*

Proof. (1) Consider a unit speed curve $s \to (x(s), 0, z(s))$ in $\mathbb{R}^3$ where $x(s) \geq 0$ and $s \geq 0$. Rotating the curve about the z-axis gives the surface of revolution

$$(x(s)\cos\phi, x(s)\sin\phi, z(s))$$

with metric $ds^2 + x(s)^2 d\phi^2$. The meridians starting at the origin are rays, so for this metric to be equal to $ds^2 + m(s)^2 d\phi^2$ we must have $m(s) = x(s)$. Since the curve has unit speed, $|x'(s)| \le 1$, so a necessary condition for writing the metric as a surface of revolution is $|m'(s)| \le 1$. It is also sufficient for if $|m'(s)| \le 1$, then we could let $z(s) := \int_0^s \sqrt{1 - (m'(s))^2}\, ds$, so that now $(m(s), z(s))$ has unit speed.

(2) If furthermore $m' > 0$ for all s, then the inverse function of $m(s)$ makes sense, and we can write the surface of revolution $(m(s)\cos\phi,\ m(s)\sin\phi,\ z(s))$ as $(x\cos\phi,\ x\sin\phi,\ g(x))$ where $x := m(s)$ and $g(x) := z(m^{-1}(x))$. Conversely, given the surface $(x\cos\phi,\ x\sin\phi,\ g(x))$, the orientation-preserving arclength parametrization $x = x(s)$ of the curve $(x, 0, g(x))$ satisfies $x' > 0$. □

Example 7.1.2. The standard $\mathbb{R}^2$ is the only von Mangoldt plane with $G_m \le 0$ that can be embedded into $\mathbb{R}^3$ as a surface or revolution because $m'(0) = 1$ and m' is non-decreasing afterwards.

Remark 7.1.3. Let M_m, not necessarily von Mangoldt, have $G_m \ge 0$. Then $m' \in [0,1]$ because $m > 0$, m' is non-increasing, and $m'(0) = 1$, so that M_m is isometric to a surface of revolution in $\mathbb{R}^3$. In fact, if $m'(s_0) = 0$, then $m|_{[s_0,\infty)} = m(s_0)$, i.e. outside the s_0-ball about the origin M_m is a cylinder. Thus except for such surfaces M_m can be written as $(x\cos\phi, x\sin\phi, g(x))$ for $g(x) = \int_0^{m^{-1}(x)} \sqrt{1 - (m'(s))^2}\, ds$. Paraboloids and two-sheeted hyperboloids are von Mangoldt planes of positive curvature [SST03, pp. 234-235] and they are of the form $(x\cos\phi, x\sin\phi, g(x))$.

Remark 7.1.4. The defining property $G'_m \le 0$ of von Mangoldt planes clearly restricts the behavior of m'. Let $Z(G_m)$ denote the set where G_m

vanishes; as M_m is von Mangoldt, $Z(G_m)$ is closed and connected, and hence it could be equal to the empty set, a point, or an interval, while m' behaves as follows.

(i) If $G_m > 0$, then m' is decreasing and takes values in $(0, 1]$.

(ii) If $G_m \le 0$, then m' is non-decreasing and takes values in $[1, \infty)$.

(iii) If $Z(G_m)$ is a positive number z, then m' decreases on $[0, z)$ and increases on (z, ∞), and m' may have two, one, or no zeros.

(iv) If $Z(G_m) = [a, b] \subset (0, \infty]$, then m' decreases on $[0, a)$, is constant on $[a, b]$, and increases on (b, ∞) if $b < \infty$. Also either $m'|_{[a,b]} = 0$ or else m' has two, or no zeros.

All the above possibilities occur with one possible exception: in cases (iii)-(iv) we are not aware of examples where m' vanishes on $Z(G_m)$.

Remark 7.1.5. Thus if M_m is von Mangoldt, then m' is monotone near infinity, so $m'(\infty)$ exists; moreover, $m'(\infty) \in [0, \infty]$, for otherwise m would vanish on $(0, \infty)$. It follows that M_m admits total curvature, which equals

$$\int_0^{2\pi} \int_0^{\infty} G_m \, m \, dr \, d\theta = -2\pi \int_0^{\infty} m'' = 2\pi(1 - m'(\infty)) \in [-\infty, 2\pi].$$

Remark 7.1.6. The zeros of m' correspond to parallels that are geodesics and are of interest. In contrast with restrictions on the zero set of m' for von Mangoldt planes, if M_m is not necessarily von Mangoldt, then any closed subset of $[0, \infty)$ that does not contain 0 can be realized as the set of zeros of m'. (Indeed, for any closed subset of a manifold there is a smooth nonnegative function that vanishes precisely on the subset [BJ82, Whitney's Theorem 14.1]. It follows that if C is a closed subset of $[0, \infty)$

that does not contain 0, then there is a smooth function $g\colon [0,\infty) \to [0,\infty)$ that is even at 0, satisfies $g(0) = 1$, and is such that $g(s) = 0$ if and only if $s \in C$. If m is the solution of $m' = g$, $m(0) = 0$; then M_m has the promised property).

A common way of constructing von Mangoldt planes involves the Jacobi initial value problem

$$m'' + Km = 0, \quad m(0) = 0, \quad m'(0) = 1 \tag{7.1.7}$$

where K is smooth on $[0,\infty)$. It follows from the proof of [KW74, Lemma 4.4] that g_m is a complete smooth Riemannian metric on $\mathbb{R}^2$ if and only if the following condition holds

($\star$) *the (unique) solution m of* (7.1.7) *is positive on* $(0,\infty)$.

Remark 7.1.8. A basic tool that produces solutions of (7.1.7) satisfying condition ($\star$) is the Sturm comparison theorem that implies that if m_1 is a positive function that solves (7.1.7) with $K = K_1$, and if K_2 is any non-increasing smooth function with $K_2 \le K_1$, then the solution m_2 of (7.1.7) with $K = K_2$ satisfies $m_2 \ge m_1$, so that g_{m_2} is a von Mangoldt plane.

Example 7.1.9. If K is a smooth function on $[0,\infty)$ such that $\max(K,0)$ has compact support, then a positive multiple of K can be realized as the curvature G_m of some M_m; of course, if K is non-increasing, then M_m is von Mangoldt. (Indeed, in [KW74, Lemma 4.3] Sturm comparison was used to show that if $\int_t^\infty \max(K,0) \le \frac{1}{4t+4}$ for all $t \ge 0$, then K satisfies ($\star$), and in particular, if $\max(K,0)$ has compact support, then there is a constant $\varepsilon > 0$ such that the above inequality holds for εK).

Remark 7.1.10. A useful addendum to Remark 7.1.8 is that the additional assumption $m_1' \geq 0$ implies $m_2' \geq m_1' > 0$. (Indeed, the function $m_1' m_2 - m_1 m_2'$ vanishes at 0 and has nonpositive derivative $(-K_1 + K_2) m_1 m_2$, so $m_1' m_2 \leq m_1 m_2'$. As m_1, m_2, m_1' are nonnegative, so is m_2'. Hence, $m_1 m_2' \leq m_2 m_2'$, which gives $m_1' m_2 \leq m_2 m_2'$, and the claim follows by canceling m_2).

7.2 Smoothed cones made von Mangoldt

Finding a von Mangoldt plane that has zero curvature (and therefore constant m') near infinity is easy, but it is harder to prescribe the value of m' there. Theorem 7.2.1 below presents what we understand on this issue.

Theorem 7.2.1. *For every $s \in (0,1]$, there exists $\rho > 0$ and a von Mangoldt plane M_m such that $m' = s$ on $[\rho, \infty)$.*

Thus, each cone in $\mathbb{R}^3$ can be smoothed to a von Mangoldt plane, but we do not know how to construct a (smooth) capped cylinder that is von Mangoldt.

Proof. We exclude the trivial case $x = 1$ in which $m(r) = r$ works.

For $u \in [0, \frac{1}{4}]$ set $K_u(r) = \frac{1}{4(r+1)^2} - u$, and let m_u be the unique solution of (7.1.7) with $K = K_u$. Then g_{m_u} is von Mangoldt. For $u > 0$ let $z_u \in [0, \infty)$ be the unique zero of K_u; note that z_u is the global minimum of m_u', and $z_u \to \infty$ as $u \to 0$.

Lemma 7.2.2. *The function $u \to m_u'(z_u)$ takes every value in $(0,1)$ as u varies in $(0, \frac{1}{4})$.*

Proof. One verifies that $m_0(r) = \ln(r+1)\sqrt{r+1}$, i.e. the right hand side solves (7.1.7) with $K = K_0$. Then $m_0' = \frac{2+\ln(r+1)}{2\sqrt{r+1}}$ is a positive function converging to zero as $r \to \infty$. By Sturm comparison $m_u \geq m_0 > 0$ and $m_u' \geq m_0' > 0$.

We now show that $m_u'(z_u) \to 0$ as $u \to +0$. To this end fix an arbitrary $\varepsilon > 0$. Fix t_ε such that $m_0'(t_\varepsilon) < \varepsilon$. By continuous dependence on parameters (m_u, m_u') converges to (m_0, m_0') uniformly on compact sets as $u \to 0$. So for all small u we have $m_u'(t_\varepsilon) < \varepsilon$ and also $t_\varepsilon < z_u$. Since m_u decreases on $(0, z_u)$, we conclude that $0 < m_u'(z_u) < m_u'(t_\varepsilon) < \varepsilon$, proving that $m_u'(z_u) \to 0$ as $u \to +0$.

On the other hand, $m_{\frac{1}{4}}'(z_{\frac{1}{4}}) = 1$ because $z_{\frac{1}{4}} = 0$ and by the initial condition $m_{\frac{1}{4}}'(0) = 1$. Finally, the assertion of the lemma follows from continuity of the map $u \to m_u'(z_u)$, because then it takes every value within $(0,1)$ as u varies in $(0, \frac{1}{4})$. (To check continuity of the map fix u_*, take an arbitrary $u \to u_*$ and note that $z_u \to z_{u_*}$, so since m_u' converges to m_{u_*}' on compact subsets, it does so on a neighborhood of z_{u_*}, so $m_u'(z_u)$ converges to $m_{u_*}'(z_{u_*})$). $\square$

Continuing the proof of the theorem, fix an arbitrary $u > 0$. The continuous function $\max(K_u, 0)$ is decreasing and smooth on $[0, z_u]$ and equal to zero on $[z_u, \infty)$. So there is a family of non-increasing smooth functions $G_{u,\varepsilon}$ depending on small parameter ε such that $G_{u,\varepsilon} = \max(K_u, 0)$ outside the ε-neighborhood of z_u. Let $m_{u,\varepsilon}$ be the unique solution of (7.1.7) with $K = G_{u,\varepsilon}$; thus $m_{u,\varepsilon}'(r) = m_{u,\varepsilon}'(z_u + \varepsilon)$ for all $r \geq z_u + \varepsilon$. If ε is small enough, then $G_{u,\varepsilon} \leq K_0$, so $m_{u,\varepsilon} \geq m_0 > 0$ and $m_{u,\varepsilon}' \geq m_0' > 0$. By continuous dependence on parameters, the function $(u, \varepsilon) \to m_{u,\varepsilon}'$ is continuous, and moreover $m_{u,\varepsilon}'(z_u + \varepsilon) \to m_u'(z_u)$ as $\varepsilon \to 0$, and u is fixed.

Fix $x \in (0,1)$. By Lemma 7.2.2 there are positive v_1, v_2 such that $m_{v_1}'(z_{v_1}) < x < m_{v_2}'(z_{v_2})$. Letting u of the previous paragraph to be v_1,

v_2, we find ε such that $m'_{v_1,\varepsilon}(z_{v_1} + \varepsilon) < x < m'_{v_2,\varepsilon}(z_{v_2} + \varepsilon)$, so by the intermediate value theorem there is u with $m'_{u,\varepsilon}(z_u + \varepsilon) = x$. Then the metric $g_{m_{u,\varepsilon}}$ has the asserted properties for $\rho = z_u + \varepsilon$. □

Chapter 8

Extending the Work of Kondo and Tanaka

In [KT10] Kondo-Tanaka generalize the Toponogov Comparison Theorem so that an arbitrary noncompact manifold M can be compared with a rotationally symmetric plane M_m (defined by the metric $dr^2 + m^2(r)d\theta^2$), and they use this to show that if M_m satisfies certain conditions, then M must be topologically finite. We substitute one of the conditions for M_m with a weaker condition and show that our method using this weaker condition enables us to draw further conclusions on the topology of M. We also completely remove one of the conditions required for the Sector Theorem, another important result by Kondo-Tanaka.

8.1 Basics

Definition 8.1.1. A manifold M is *topologically finite* if it is homeomorphic to the interior of a compact set with boundary.

Definition 8.1.2. Let (M, p) denote a manifold with arbitrary basepoint $p \in M$, let M_m denote a rotationally symmetric plane with origin o, let G be the curvature function for M, and for any meridian $\mu(t)$ emanating from $o = \mu(0)$, let $G_m(\mu(t))$ be the curvature at $\mu(t)$. We say that (M,p) has *radial curvature bounded below* by that of M_m if, along every unit-speed minimal geodesic $\gamma : [0,a) \to M$ emanating from $p = \gamma(0)$, we have $G(\sigma_t) \geq G_m(\mu(t))$ for all $t \in [0,a)$ and all 2-dimensional subspaces σ_t spanned by $\gamma'(t)$ and an element of $T_{\gamma(t)}M$.

Definition 8.1.3. Given (M,p), a point $q \in M$ is a *critical point of* $d(\cdot,p)$ if, given any $v \in T_qM$, there exists a minimal geodesic γ emanating from q to p such that $\measuredangle(\dot\gamma(0), v) \leq \frac{\pi}{2}$.

Definition 8.1.4. We say that M_m is a *Cartan-Hadamard plane* if $G_m \leq 0$ everywhere.

The critical point theory of distance functions by Grove-Shiohama [GrSh], [Gro93], [Gre97, Lemma 3.1], [Pet06, Section 11.1] implies the following *Isotopy Lemma* [Pet06, Section 11.1]:

Theorem 8.1.5. (Isotopy Lemma) *Given (M, p), suppose that for* R_1, R_2 *with* $0 < R_1 < R_2 \leq \infty$, $\overline{B_{R_2}(p)} \setminus B_{R_1}(p)$ *has no critical point of* $d(\cdot,p)$. *Then* $\overline{B_{R_2}(p)} \setminus B_{R_1}(p)$ *is homeomorphic to* $\partial B_{R_1}(p) \times [R_1, R_2]$.

Remark 8.1.6. Theorem 8.1.5 implies that M is topologically finite if the set of critical points of $d(\cdot,p)$ is confined to a subset of finite radius.

Definition 8.1.7. Given a rotationally symmetric plane M_m, we define a *sector of angular measure* δ, $V(\delta)$, as

$$V(\delta) := \{q \in M_m | 0 < \theta(q) < \delta\}$$

Likewise we define a *closed sector of angular measure* δ, $\overline{V}(\delta)$, as

$$\overline{V}(\delta) := \{q \in M_m | 0 \leq \theta(q) \leq \delta\}$$

Definition 8.1.8. When we say that a sector $V(\delta)$ or $\overline{V}(\delta)$ is *free of cut points* or is *cut-point-free*, we mean that there does not exist a pair of points q, q' in the sector such that if γ is a minimal geodesic joining q to q', q' is a cut point of q. For example, if M_m is von Mangoldt, $V(\pi)$ is free of cut points.

8.2 The Generalized Toponogov Comparison Theorem

Remark 8.2.1. The main result in [KT10], which we improve on, is founded on a generalized version of the Toponogov Comparison Theorem (Theorem 8.2.2). We present here a brief history leading up to this generalized version.

Let M_k denote a 2-dimensional manifold with curvature $\geq k$ and S_k the comparison space with constant curvature k. Also let $\triangle(M_k)$ denote a triangle in M_k and $\triangle(S_k)$ a comparison triangle of S_k with corresponding sides of the same length. In 1955, A. D. Alexandrov [Al] proved that in this setting, the angles of $\triangle(M_k)$ are greater than or equal to the corresponding angles of $\triangle(S_k)$. In 1959, V. A. Toponogov [To1], [To2] improved on Alexandrov's results so that M_k can have any dimension ≥ 2; this work is the widely known Toponogov Comparison Theorem. In 1980, D. Elerath [Ele80] proved the above inequality for a triangle in M_k with $k \geq 0$ and a comparison triangle in a von Mangoldt plane embedded in $\mathbb{R}^3$. In 1985, U. Abresch [A] developed a way of using a Cartan-Hadamard plane as a comparison space. In 2003, Y. Itokawa, Y. Machigashira, and K. Shiohama [IMS03] improved on Elerath's results so that the comparison von Mangoldt plane does not have to be embeddable in $\mathbb{R}^3$ and so that the angle inequality applies to all three pairs of corresponding angles (in

Elerath's work the inequality applies to only two of the pairs). Finally, in 2010, K. Kondo and M. Tanaka [KT10] generalized the Toponogov Comparison Theorem in the following way:

Theorem 8.2.2. *Let the radial curvature of* (M, p) *be bounded below by that of* M_m. *Assume that* M_m *admits a sector* $V(\delta)$ *for some* $\delta \in (0, \pi)$ *that has no pair of cut points. Then, for every geodesic triangle* $\triangle(pxy)$ *in* M *with* $\measuredangle(xpy) < \delta$, *there exists a geodesic triangle* $\triangle(\tilde{p}\tilde{x}\tilde{y})$ *in* $V(\delta)$ *such that*

$$d(\tilde{p}, \tilde{x}) = d(p, x), \quad d(\tilde{p}, \tilde{y}) = d(p, y), \quad d(\tilde{x}, \tilde{y}) = d(x, y)$$

and that

$$\measuredangle(xpy) \geq \measuredangle(\tilde{x}\tilde{p}\tilde{y}), \quad \measuredangle(pxy) \geq \measuredangle(\tilde{p}\tilde{x}\tilde{y}), \quad \measuredangle(pyx) \geq \measuredangle(\tilde{p}\tilde{y}\tilde{x}).$$

Remark 8.2.3. In the original Toponogov Comparison Theorem, the requirement of curvature bounding from below is the same, but no basepoint is needed because constant curvature spaces are homogeneous.

Remark 8.2.4. The lemma below is key to proving the Generalized Toponogov Theorem in [KT10]. We state it in full because we also use it to prove one of our results.

Lemma 8.2.5. ([Lemma 4.11, [KT10]) *Let the radial curvature of* (M, p) *be bounded below by that of* M_m. *Assume that* M_m *admits a sector* $V(\delta)$ *for some* $\delta \in (0, \pi)$ *that has no pair of cut points. If a geodesic triangle* $\triangle pxy$ *in* M_m *admits a geodesic triangle* $\triangle\tilde{p}\tilde{x}\tilde{y}$ *in* $V(\delta)$ *satisfying*

$$d(\tilde{p}, \tilde{x}) = d(p, x), \quad d(\tilde{p}, \tilde{y}) = d(p, y), \quad d(\tilde{x}, \tilde{y}) = d(x, y),$$

then

$$\measuredangle(pxy) \geq \measuredangle(\tilde{p}\tilde{x}\tilde{y}) \text{ and } \measuredangle(pyx) \geq \measuredangle(\tilde{p}\tilde{y}\tilde{x}).$$

8.3 The Two Theorems

Below are the two results in [KT10] that we extend.

Theorem 8.3.1. (Main Theorem) *Let M be a complete open Riemannian n-manifold whose radial curvature at the basepoint p is bounded below by that of a noncompact rotationally symmetric plane M_m. Assume that there exists a sector $V(\delta)$ in M_m that does not contain a pair of cut points. Also suppose M_m has finite total curvature. Then M has finite topological type.*

Remark 8.3.2. Since the main theorem calls for a rotationally symmetric plane with a cut-point-free sector, it is natural to wonder what surfaces satisfy this criterion. The Sector Theorem gives two such classes of planes.

Theorem 8.3.3. (Sector Theorem) *Let M_m be a noncompact rotationally symmetric plane that is von Mangoldt or Cartan-Hadamard outside a ball of finite radius about o. If M_m admits finite total curvature, then there exists $\delta \in (0, \pi)$ such that $V(\delta)$ has no pair of cut points.*

8.4 Extending the Main Theorem

We modify Theorem 8.3.1 by replacing the condition of finite curvature with the condition that $m'(r)$ be bounded. Note that bounded $m'(r)$ is more general than $c(M_m) > -\infty$. Indeed, if M_m admits total curvature, then we have

$$c(M_m) = \int_0^{2\pi} \int_0^{\infty} G_m(r) m(r) dr d\theta = -2\pi \int_0^{\infty} m'' = 2\pi(1 - m'(\infty)) \in [-\infty, 2\pi]$$

So, $c(M_m) > -\infty$ implies $m'(\infty) \in [0, \infty)$. Hence, $m'(r)$ must be bounded on all r.

On the other hand, there exists a rotationally symmetric plane such that total curvature is not admitted but $m'(r)$ is bounded on all r: define $m(r)$ as $m(r) = r$ on $[0, 2\pi]$ and $m(r) = r - \frac{1}{2}\sin r$ on $(2\pi, \infty)$. Next, smooth out $m(r)$ on a neighborhood σ of 2π such that $m(r) > 0$ on σ. Then $m(r)$ is a smooth function on $[0, \infty)$ that can be extended to a smooth odd function around 0 with $m(r) > 0$ for all r, $m(0) = 0$, and $m'(0) = 1$. Hence the metric $dr^2 + m^2(r)d\theta^2$ describes a rotationally symmetric plane. Since $m'(r) = 1 - \frac{1}{2}\cos r$ does not converge to a limit as $r \to \infty$, M_m does not admit total curvature. However, $m'(r) = 1 - \frac{1}{2}\cos r$ is bounded on all r.

Convention: From this point on, set $N := \sup\{m'(r)\}$.

Remark 8.4.1. Since $m'(0) = 1$ for any M_m, we have $N \geq 1$ always. Also, note that M_m is isometric to $\mathbb{R}^2$ if and only if $m'(r)$ is identically 1.

Lemma 8.4.2. *Let M_m be a rotationally symmetric plane with metric $dr^2 + m^2(r)d\theta^2$, and let $N < \infty$. Then $\gamma_q : [0, \infty) \to M_m$ has turn angle $\geq \frac{\pi}{2N}$. Furthermore, if M_m is not isometric to $\mathbb{R}^2$, then $\gamma_q : [0, \infty) \to M_m$ has turn angle $> \frac{\pi}{2N}$.*

Proof. If γ_q is not an escaping geodesic, then it must have infinite turn angle by Lemma 4.3.3. So assume γ_q is escaping. Let c be the Clairaut constant of γ_q, and let ρ be the value at which $N\rho = c = m(r_q)$. Since $N \geq m'(r)$ for all r, we have

$$\int_0^r N dr = Nr \geq m(r) = \int_0^r m'(r) dr$$

for any r.

This implies

$$T_{\gamma_q} = \int_{r_q}^{\infty} \frac{cdr}{m(r)\sqrt{m^2(r) - c^2}} \geq \int_{\rho}^{\infty} \frac{cdr}{Nr\sqrt{(Nr)^2 - c^2}}.$$

Now we show that the second integral equals $\frac{\pi}{2N}$. Applying the change of variables $r := \frac{ct}{N}$, we have

$$\int_1^{\infty} \frac{c\frac{c}{N}dt}{ct\sqrt{(ct)^2 - c^2}} = \int_1^{\infty} \frac{dt}{N\,t\,\sqrt{t^2-1}} = -\frac{1}{N}\text{arccot}(\sqrt{t^2-1})|_1^{\infty} = \frac{\pi}{2N}.$$

It follows trivially that if M_m is not isometric to $\mathbb{R}^2$, then $N > 1$ and $m' < N$ for some r, so $T_{\gamma_q} > \frac{\pi}{2N}$.

□

Lemma 8.4.3. *Let M_m be such that there exists a sector $V(\delta)$ free of cut points and $N < \infty$. If σ is a ray with $\kappa_\sigma \geq \frac{\pi}{2}$, then $T_\sigma \geq \min(\frac{\pi}{2N}, \delta)$. If, furthermore, M_m is not isometric to $\mathbb{R}^2$ and if $\delta > \frac{\pi}{2N}$, then $T_\sigma > \frac{\pi}{2N}$.*

Proof. If γ_q is not escaping, then it has infinite turn angle by Lemma 4.3.3. If γ_q is escaping, then $T_{\gamma_q} \geq \frac{\pi}{2N}$ by Lemma 8.4.2. Choose $\epsilon < \min(\frac{\pi}{2N}, \delta)$ and assume $q \in \partial\overline{V}(\epsilon)$. Now γ_q and $\overline{V}(\epsilon)$ determine a bounded region. For small $t > 0$, because $\kappa_\sigma \geq \frac{\pi}{2}$, $\sigma(t)$ lies in this region. In order for σ to escape this region, either $T_\sigma > \epsilon$ or it must intersect γ_q within $\overline{V}(\epsilon)$. But the latter is impossible, so $T_\sigma > \epsilon$. Since ϵ was arbitrary, we have $T_\sigma \geq \min(\frac{\pi}{2N}, \delta)$.

Suppose M_m is not isometric to $\mathbb{R}^2$ and $\delta > \frac{\pi}{2N}$. Even if γ_q is escaping, $T_{\gamma_q} > \frac{\pi}{2N}$ by Lemma 8.4.2. Hence, γ_q and $\overline{V}(\frac{\pi}{2N})$ determine a bounded region, and for small $t > 0$, because $\kappa_\sigma \geq \frac{\pi}{2}$, $\sigma(t)$ lies in this region. In order for σ to escape this region, either $T_\sigma > \frac{\pi}{2N}$ or it must intersect γ_q within $\overline{V}(\frac{\pi}{2N})$. But the latter is impossible.

□

Lemma 8.4.4. *Let the radial curvature of* (M, p) *be bounded below by that of* M_m *with a cut-point-free sector* $V(\delta)$*, let* q *be a critical point of* $d(\cdot, p)$*, and let* $\gamma : [0, \infty) \to M$ *be a ray emanating from* p*. Let* α *be a minimal geodesic connecting* $p = \alpha(0)$ *to* q *such that* $\measuredangle(\dot{\gamma}(0), \dot{\alpha}(0)) =: \theta < \delta$*. Then there exists a ray* $\tilde{\eta} \subset M_m$ *with* $T_{\tilde{\eta}} \leq \theta$ *and* $\kappa_{\tilde{\eta}} \geq \frac{\pi}{2}$*.*

Proof. If q is a critical point of $d(\cdot, p)$, then we can always construct a triangle $\subset M$ with q a vertex and one of the sides $\subset \gamma$, since γ cannot pass through q; indeed, if it did, then $\gamma|_{[0,d(p,q)]}$ would be the only minimal geodesic joining q to p, which is impossible since q is a critical point of $d(\cdot, p)$.

Let η_j be a minimal geodesic joining q to $\gamma(t_j)$, where $t_j \to \infty$ as $j \to \infty$. Consider the sequence of triangles $\triangle(pq\gamma(t_j))$, consisting of edges α, η_j, and $\gamma|_{[0,t_j]}$. Since $\measuredangle(qp\gamma(t_j)) = \theta$ for each j, the generalized Toponogov theorem implies that there exists a sequence of comparison triangles $\triangle\tilde{p}\tilde{q}\tilde{\gamma}(t_j) \subset M_m$ with corresponding sides (all minimal geodesics) of equal length and corresponding angles dominated by those in $\triangle pq\gamma(t_j)$. In particular, $\triangle\tilde{p}\tilde{q}\tilde{\gamma}(t_j) \subset \overline{V}(\theta)$.

Since $\ell(\eta_j) \to \infty$ as $j \to \infty$, we have $\ell(\tilde{\eta}_j) \to \infty$ as $j \to \infty$. Hence $\{\tilde{\eta}_j\}$ must subconverge to a ray $\tilde{\eta}$. Since $T_{\tilde{\eta}_j} \leq \theta$ for each j, we have $T_{\tilde{\eta}} \leq \theta$.

Since q is a critical point of $d(\cdot, p)$, there exists a minimal geodesic σ emanating from p to q such that $\measuredangle(-\dot{\sigma}(d(p,q)), \dot{\eta}_j(0)) \leq \frac{\pi}{2}$. Let $\triangle p\sigma(d(p,q))\gamma(t_j)$ denote the triangle consisting of the edges σ, η_j, and $\gamma|_{[0,t_j]}$. Since $\triangle p\sigma(d(p,q))\gamma(t_j)$ has the same side lengths as $\triangle pq\gamma(t_j)$ (with edges α, η_j, and $\gamma|_{[0,t_j]}$), it admits the triangle $\triangle\tilde{p}\tilde{q}\tilde{\gamma}(t_j)$ satisfying the angle inequalities in Lemma 8.2.5. In particular, $\measuredangle(\tilde{p}\tilde{q}\tilde{\gamma}(t_j)) \leq \measuredangle(-\dot{\sigma}(d(p,q)), \dot{\eta}_j(0)) \leq \frac{\pi}{2}$. Since the segment joining $\tilde{p}$ to $\tilde{q}$ is a subarc of a meridian, we have $\kappa_{\tilde{\eta}_j} \geq \frac{\pi}{2}$ for each j. Hence, in the limit, $\kappa_{\tilde{\eta}} \geq \frac{\pi}{2}$.

□

Lemma 8.4.5. *Let the radial curvature of (M,p) be bounded below by that of M_m with $V(\delta)$ free of cut points and $N<\infty$, let q be a critical point of $d(\cdot,p)$, let γ be a ray emanating from p, and let α be a minimal geodesic joining $p=\alpha(0)$ to q. Then $\measuredangle(\dot\gamma(0),\dot\alpha(0)) \geq \min(\frac{\pi}{2N},\delta)$. Furthermore, if M_m is not isometric to $\mathbb{R}^2$ and if $\delta > \frac{\pi}{2N}$, then $\measuredangle(\dot\gamma(0),\dot\alpha(0)) > \frac{\pi}{2N}$.*

Proof. Suppose $\measuredangle(\dot\gamma(0),\dot\alpha(0)) < \min(\frac{\pi}{2N},\delta)$. Lemma 8.4.4 implies that there exists a ray $\tilde\eta \subset M_m$ with $T_{\tilde\eta} < \min(\frac{\pi}{2N},\delta)$ and $\kappa_{\tilde\eta} \geq \frac{\pi}{2N}$. But Lemma 8.4.3 implies $T_{\tilde\eta} \geq \min(\frac{\pi}{2N},\delta)$, a contradiction.

Now suppose M_m is not isometric to $\mathbb{R}^2$ and $\delta > \frac{\pi}{2N}$, and assume $\measuredangle(\dot\gamma(0),\dot\alpha(0)) \leq \frac{\pi}{2N}$. Lemma 8.4.4 implies that there exists a ray $\tilde\eta \subset M_m$ with $T_{\tilde\eta} \leq \frac{\pi}{2N}$. But Lemma 8.4.3 implies $T_{\tilde\eta} > \frac{\pi}{2N}$, a contradiction.

□

Theorem 8.4.6. *Let the radial curvature of (M,p) be bounded below by that of M_m with $N<\infty$ and $V(\delta)$ free of cut points. Then M is topologically finite.*

Proof. We prove the claim by showing that $\{q_i\}$, the set of critical points of $d(\cdot,p)$, is bounded. Suppose the set is unbounded. Let α_i be a minimal geodesic emanating from p to q_i. Since $\ell(\alpha_i)\to\infty$, $\{\alpha_i\}$ must subconverge to a ray γ emanating from p. In particular, there exists α such that $\measuredangle(\dot\gamma(0),\dot\alpha(0)) < \min(\delta,\frac{\pi}{2N})$. But this is impossible by Lemma 8.4.5.

□

Theorem 8.4.7. *Let the radial curvature of (M,p) be bounded below by that of M_m containing a cut-point-free sector $V(\delta)$ with $\delta > \frac{\pi}{2}$. Suppose M_m is not isometric to $\mathbb{R}^2$ and $N=1$. If p is a critical point of infinity, then M is homeomorphic to $\mathbb{R}^n$, where n is the dimension of M.*

Proof. We prove the claim by showing that M has no critical point of $d(\cdot, p)$. Suppose q were a critical point of $d(\cdot, p)$, and let α be a minimal geodesic joining q to p. For any ray γ emanating from p, we must have $\measuredangle(\dot\alpha(0), \dot\gamma(0)) > \frac{\pi}{2N} = \frac{\pi}{2}$ by Lemma 8.4.5. But since p is a critical point of infinity, $\measuredangle(\dot\alpha(0), \dot\gamma(0))$ is $\leq \frac{\pi}{2}$ for some ray γ emanating from p, a contradiction. □

Remark 8.4.8. If M_m is a von Mangoldt plane of nonnegative curvature not isometric to $\mathbb{R}^2$, then it satisfies the conditions for Theorem 8.4.7.

Remark 8.4.9. Let the radial curvature of (M, p) be bounded below by that of a von Mangoldt plane M_m with nonnegative curvature. Let r, r_m denote the distance functions to the basepoints p, o of M, M_m, respectively. Let $R := \sup\{r_m(\mathfrak{C}_m)\}$; by Theorem 5.1.1, $R < \infty$ if and only if $m'(\infty) < \frac{1}{2}$. Proposition 8.4.10 below, the Isotopy Lemma, and Theorem 5.1.1 imply that if $R < \infty$, R can be explicitly determined, M is topologically finite, and R is an upper bound on the radius of the set $S \subset M$ that determines the topology of M.

Proposition 8.4.10. *Let the radial curvature of (M, p) be bounded below by that of a von Mangoldt plane M_m. Let r, r_m denote the distance functions to the basepoints p, o of M, M_m, respectively. If q is a critical point of r, then $r(q)$ is contained in $r_m(\mathfrak{C}_m)$.*

Proof. Assuming $r(q) \notin r_m(\mathfrak{C}_m)$ we will show that q is not a critical point of r. Since M is complete and noncompact, there exists a ray γ emanating from q. Consider the comparison triangle $\triangle o, \tilde q, \tilde\gamma(t_i)$ in M_m for any geodesic triangle with vertices p, q, $\gamma(t_i)$. Passing to a subsequence, arrange so that the segments $[\tilde q, \tilde\gamma(t_i)]$ subconverge to a ray, which we denote by $\tilde\gamma$. Since $\tilde q \notin \mathfrak{C}_m$, the angle formed by $\tilde\gamma$ and $[o, \tilde q]$ is $> \frac{\pi}{2}$, and hence for large t_i the same is true for the angles formed by

$[q, \gamma(t_i)]$ and $[p, q]$. By comparison, γ forms angle $> \frac{\pi}{2}$ with any segment joining q to p, i.e. q is not a critical point of r. □

8.5 Improving on the Sector Theorem

In the Sector Theorem, the condition of finite total curvature can be dropped.

Convention: For all geodesic segments $\gamma : [o, \ell] \to M_m$, assume $r_{\gamma(\ell)} \geq r_{\gamma(0)}$.

Lemma 8.5.1. (Lemma 3.1, [KT10]) *Given M_m, let $V_i := V(\frac{1}{i})$ for each $i = 1, 2, ...$ Assume that there exist a constant $r_0 > 0$ and a sequence $\{\sigma_i : [0, \ell_i] \to V_i\}$ of geodesic segments such that $\sigma_i([0, \ell_i]) \bigcap \overline{B_{r_0}(o)} \neq \emptyset$ for each i and that $\liminf_{i\to\infty} r(\sigma_i(\ell_i)) > r_0$. Then, $\lim_{i\to\infty} c_i = 0$ holds, where c_i denotes the Clairaut constant of σ_i.*

Lemma 8.5.2 below combines parts of Propositions 7.2.1 and 7.2.2 in [SST, p. 220].

Lemma 8.5.2. (Propositions 7.2.1, 7.2.2, [SST03]) *Given $q \in M_m$, let $\gamma : [0, s] \to M$, $\gamma(0) = q$ be a geodesic not tangent to the parallel or meridian through q. If $\dot{r}_\gamma$ is nonzero on $[0, s)$, then there exists a Jacobi field $X(t)$ along γ that can be expressed as*

$$X(t) = sign\left(\frac{\pi}{2} - \kappa_\gamma\right) \dot{r}(t) \int_{d(o,q)}^{r(t)} \frac{m(r)}{\sqrt{m^2(r) - c^2}^3} dr \left\{ -c \frac{\partial}{\partial r_{\gamma(t)}} + \dot{r}(t) \frac{\partial}{\partial \theta_{\gamma(t)}} \right\}$$

on $[0, s)$, where c is the Clairaut constant of γ.

Lemma 8.5.3. *Given $q \in M_m$, let $\gamma : [0, s] \to M_m$, $g(0) = q$ be a geodesic that is not tangent to the parallel or meridian through q. If $\dot{r}_\gamma$ is nonzero on $[0, s)$, then there exists no conjugate point of q along $\gamma|_{[0,s)}$.*

Proof. Each additive term in the expression for $X(t)$ in Lemma 8.5.2 carries $\dot{r}(t)$. Hence, $\dot{r}(t)$ nonzero on $[0,s)$ implies that the Jacobi field $X(t)$ is nonzero on $[0,s)$. □

Lemma 8.5.4 makes our modification of [KT10, Key Lemma] possible.

Lemma 8.5.4. *Let M_m be such that $\liminf_{r\to\infty} m(r) > 0$. Let $\{\sigma_i : [0,\ell_i] \to M_m\}$ be a sequence of minimal geodesics such that $\ell_i \to \infty$, $c_i \neq 0$, and $c_i \to 0$. Then there exists $L > 0$ such that for all $i \geq L$, there does not exist any value t at which both $\dot{r}_{\sigma_i(t)} = 0$ and $\ddot{r}_{\sigma(t)} < 0$ hold.*

Proof. By contradiction; suppose that for any $L > 0$, there exists $i \geq L$ such that $\dot{r}_{\sigma_i(t_i)} = 0$ and $\ddot{r}_{\sigma_i(t_i)} < 0$ for some t_i. Choose such a subsequence and denote it $\{\sigma_i\}$. By reflectional symmetry and uniqueness of geodesics, r_{σ_i} attains its absolute maximum at t_i. Since $c_i = m(r_{\sigma_i(t_i)})$, $c_i \to 0$, we have $m(r_{\sigma_i(t_i)}) \to 0$. Since $\liminf_{r\to\infty} m(r) > 0$, $m(r_{\sigma(t_i)}) \to 0$ implies $r_{\sigma_i(t_i)} \to 0$. But this is impossible, since $\ell_i \to \infty$ and σ_i is a minimal geodesic.

□

Definition 8.5.5. Given any $q \in M$, M a complete Riemannian manifold, we define the *segment domain* of q as

$$\{v \in T_qM \mid \exp_q tv : [0,1] \to M \text{ is a minimal geodesic}\}$$

Remark 8.5.6. It is well known that the segment domain of any $q \in M$ is star-shaped and closed. The *interior* of the segment domain of q, denoted $I(q)$, is likewise defined as

$$\{v \in T_qM \mid \exp_q tv : [0,1) \to M \text{ is a minimal geodesic}\}$$

Note that $\exp_q$ is one-to-one on $I(q)$, so if x is in the image of $I(q)$, denoted $I(q)^*$, there exists a unique minimizing geodesic γ connecting q to x, and there exists $\epsilon > 0$ such that γ minimizes on $(0, d(q,x)+\epsilon)$. Hence, if x is conjugate to q, x cannot be in $I(q)^*$.

Lemma 8.5.7. *Let $\{\sigma_i : [0,\ell_i] \to M_m\}$ be a sequence of minimal geodesics converging to $\sigma : [0,\ell] \to M_m$, where σ is a subarc of a meridian. For all i large enough, $\sigma_i(\ell_i)$ is in $I(\sigma_i(0))^*$ and $\sigma_i(0)$ is in $I(\sigma_i(\ell_i))^*$.*

Proof. Since any subarc of a meridian is distance-minimizing, $\sigma(\ell)$ is in $I(\sigma(0))^*$. Hence for i large enough, $\sigma_i(\ell_i)$ is also in $I(\sigma(0))^*$. It follows that $\sigma(0)$ is in $I(\sigma_i(\ell_i))^*$, since the above implies that $\sigma(0)$ is joined to $\sigma_i(\ell_i)$ by a unique minimal geodesic and $\sigma(0)$ cannot be conjugate to $\sigma_i(\ell_i)$. So for i large enough, $\sigma_i(0)$ is in $I(\sigma_i(\ell_i))^*$. It must also follow that $\sigma_i(\ell_i)$ is in $I(\sigma_i(0))^*$.

□

Remark 8.5.8. Below we give the original version of [KT10, Key Lemma], followed by our modified version and its proof. The proof of our modified version is closely modeled on that of the original version.

Lemma 8.5.9. (Key Lemma, [KT10]) *Let M_m have finite total curvature. For each $r > 0$, there exists a number $\delta(r) \in (0,\pi)$ such that $\sigma([0,\ell]) \cap \overline{B_r(o)} = \emptyset$ holds for any minimal geodesic segment $\sigma : [0,\ell] \to V(\delta(r)) \subset M$, along which $\sigma(0)$ is conjugate to $\sigma(\ell)$.*

Lemma 8.5.10. (Modified Key Lemma) *Let M_m be such that* $\liminf_{r\to\infty} m(r) > 0$. *For each $r > 0$, there exists a number $\delta(r) \in (0,\pi)$ such that $\sigma([0,\ell]) \cap \overline{B_r(o)} = \emptyset$ holds for any minimal geodesic segment $\sigma : [0,\ell] \to V(\delta(r)) \subset M$, along which $\sigma(0)$ is conjugate to $\sigma(\ell)$.*

Proof. By contradiction. To establish the existence of $\delta(r) \in (0, \pi)$, all we need to do is show that there exists $\delta(r) > 0$, since we have $|\theta(\sigma(0)) - \theta(\sigma(\ell))| < \pi$ for any minimal geodesic segment $\sigma : [0, \ell] \to M \setminus \{o\}$. Put $V_i := V(\frac{1}{i})$ for each i. Assume that there exists a constant $r_0 > 0$ and a sequence of minimal geodesic segments $\{\sigma_i : [0, \ell_i] \to V_i\}$, with $\sigma_i(0)$ conjugate to $\sigma_i(\ell_i)$ along σ_i, such that $\sigma_i([0, \ell_i]) \cap \overline{B_{r_0}(o)} \neq \emptyset$ for each i.

We want to establish that the sequence of Clairaut constants, $\{c_i\}$, converges to 0 as $i \to \infty$. We do this by showing that $\lim_{i\to\infty} \ell_i = \infty$; indeed, this implies $\liminf_{i\to\infty} r_{\sigma_i(l_i)} > r_0$, whereupon by Lemma 8.5.1 $\{c_i\} \to 0$.

Suppose $\lim_{i\to\infty} \ell_i < \infty$ or does not exist. Then there exists $M < \infty$ such that given any N, there exists $i \geq N$ such that $\ell_i \leq M$. Then we have a subsequence of $\{\sigma_i\}$ such that the endpoints $\{\sigma_i(0)\}$, $\{\sigma_i(\ell_i)\}$ are confined to a compact set. Let $\{\sigma_i\}$ denote this subsequence. Since each σ_i is a minimal geodesic, $\{\sigma_i\}$ must lie in a bounded set. By the Arzela-Ascoli theorem, there exists a geodesic σ to which some subsequence $\{\sigma_{i_j}\}$ converges, and by construction σ must be a subarc of a meridian. Let $\sigma(0)$ be the point to which $\{\sigma_{i_j}(0)\}$ converges and let $\sigma(\ell)$ be the point to which $\{\sigma_{i_j}(\ell_{i_j})\}$ converges. For j large enough, $\sigma_{i_j}(0)$ is in $I(\sigma_{i_j}(\ell_{i_j}))^*$ and $\sigma_{i_j}(\ell_{i_j})$ is in $I(\sigma_{i_j}(0))^*$ by Lemma 8.5.7. Remark 8.5.6 implies that $\sigma_{i_j}(0)$ cannot be conjugate to $\sigma_{i_j}(\ell_{i_j})$, a contradiction. Hence we establish that $\liminf_{i\to\infty} r_{\sigma_i(\ell_i)} > r_0$.

Since $\sigma_i(0)$ and $\sigma_i(\ell_i)$ are conjugate, there exists a positive parameter value a_i at which $\dot{r}_{\sigma_i} = 0$ by Lemma 8.5.3. From our work above, we have $c_i \to 0$ and $\ell_i \to \infty$, and by assumption $\liminf_{r\to\infty} m(r) > 0$, so by Lemma 8.5.4, there exists J such that for all $i > J$, we cannot have $\ddot{r}_{\sigma_i}(a_i) < 0$. From this point on, assume $i > J$ always. Since σ_i is tangent to a parallel from above, $r_{\sigma_i(a_i)}$ is the absolute minimum of r_{σ_i}, implying $r_{\sigma_i(a_i)} \in B_{r_0}(o)$.

Let $u_i \in [a_i, \ell_i]$ be a parameter value of σ_i such that $r_{\sigma_i(u_i)} = r_0$. Set $\triangle_i :=$ the triangle $o\sigma_i(a_i)\sigma_i(u_i)$. This triangle lies in $\overline{B_{r_0}(o)} \bigcap V_i$. The angle at $\sigma_i(a_i)$ equals $\frac{\pi}{2}$ by construction. The angle at $o < \frac{1}{i}$, so it tends to 0 as $i \to \infty$. This implies that the area of $\triangle_i$ tends to 0 as $i \to \infty$.

Now consider the angle at $\sigma(u_i)$. On the one hand, since $c_i \to 0$, the angle at $\sigma(u_i)$ must go to 0. On the other hand, the curvature function $G_m(r)$ attains its maximum and minimum on $[0, r_0]$, so $\int_{\triangle_i} G_m \to 0$ as $i \to \infty$. The Gauss-Bonnet theorem gives $\{$ sum of the interior angles $\} = \pi + \int_{\triangle_i} G_m$, so we have $\{$ sum of the interior angles $\} \to \pi$ as $i \to \infty$. This means that the angle at $\sigma_i(u_i)$ must approach $\frac{\pi}{2}$ as $i \to \infty$, a contradiction.

□

Lemma 8.5.11. *Suppose M_m is a noncompact complete rotationally symmetric plane that is von Mangoldt or Cartan-Hadamard outside a compact set. If $\liminf_{r\to\infty} m(r) = 0$, then M_m has finite total curvature.*

Proof. We prove our claim by showing that $\lim_{r\to\infty} m'(r)$ exists and is finite.

Let $R > 0$ be such that M_m is von Mangoldt or Cartan-Hadamard on $M_m \setminus \overline{B_R(o)}$. There exists $r_0 > R$ at which $m' < 0$, for if $m'(r) \geq 0$ for all $r > R$, then $\liminf_{r\to\infty} m(r) > 0$. Because $m(r) > 0$ on $r > 0$, we cannot have $m'(r) \leq m'(r_0)$ on $[r_0, \infty)$. Hence there exists $r_1 > r_0$ such that $m'(r_1) < 0$ and $m''(r_1) > 0$. Also $G_m(r_1) < 0$. Since M_m is von Mangoldt or Cartan-Hadamard on (R, ∞), $G_m(r) \leq 0$ on $[r_1, \infty)$, implying $m''(r) \geq 0$ on $[r_1, \infty)$.

We claim $m' < 0$ on $[r_1, \infty)$. Indeed, if for some $r \geq r_1$ $m' \geq 0$, then $m''[r_1, \infty) \geq 0$ implies $m' \geq 0$ for all $r \geq r_1$, implying $\liminf_{r\to\infty} m(r) > 0$.

Since m' is an increasing function on $[r_1, \infty)$ that is bounded above by 0, it must converge to a finite number.

□

Lemma 8.5.12. *Let M_m be von Mangoldt or Cartan-Hadamard outside a compact set. Then for each $r > 0$, there exists a constant number $\delta(r) \in (0, \pi)$ such that $\sigma([0, \ell]) \cap \overline{B_r(o)} = \emptyset$ holds for any minimal geodesic segment $\sigma : [0, \ell] \to V(\delta(r)) \subset M$, along which $\sigma(0)$ is conjugate to $\sigma(\ell)$.*

Proof. Either $\liminf_{r\to\infty} m(r) > 0$ or $\liminf_{r\to\infty} m(r) = 0$. If $\liminf_{r\to\infty} m(r) > 0$, then the claim holds by Lemma 8.5.10. If $\liminf_{r\to\infty} m(r) = 0$, then Lemma 8.5.11 applies, so M_m has finite total curvature. Lemma 8.5.9 (the original version of the Key Lemma) then implies the claim. □

Remark 8.5.13. Below we give the statement and proof of the improved Sector Theorem. The basic reasoning is identical to its counterpart in [KT10] except that references to the Key Lemma are replaced by references to Lemma 8.5.12.

Theorem 8.5.14. (Improved Sector Theorem) *Let M_m be von Mangoldt or Cartan-Hadamard outside a compact set. Then M_m has a sector with no pair of cut points.*

Proof. Let M_m be von Mangoldt or Cartan-Hadamard outside $\overline{B_{R_0}(o)}$ for some $R_0 > 0$. Fix any $R_1 > R_0$, and in the setting of Lemma 8.5.12, let $\delta(R_1) \in (0, \pi)$ be the number such that if $\sigma : [0, \ell] \to V(\delta(R_1))$ is a minimal geodesic along which $\sigma(0)$ is conjugate to $\sigma(\ell)$, then

$$\sigma[0, \ell] \cap \overline{B_{R_1}(o)} = \emptyset.$$

Proceeding by contradiction, suppose $q \in V(\delta(R_1))$ has a cut point $x \in V(\delta(R_1))$. We will show that there exists a point conjugate to q in

$V(\delta(R_1))$. If x is conjugate to q, we are done, so suppose not. Then let α, β be minimal geodesics connecting q to x and bounding a region D. The boundary of D only meets C_q at x because α, β are minimal. By assumption x is in C_q but is not conjugate to q, so there exists a geodesic in D emanating from q and meeting C_q in the interior of D; that is, the interior of D meets C_q. Since C_q is a tree by Lemma 2.5.9, the interior of D contains an endpoint of C_q, which is conjugate to q. So from this point on, assume q is conjugate to $x \in V(\delta(R_1))$ along a minimal geodesic γ_x.

Now we derive our contradictions. Suppose $M_m \setminus \overline{B_{R_1}(o)}$ is Cartan-Hadamard. By Lemma 8.5.12, γ_x or any geodesic γ' emanating from q that is close enough to γ_x does not intersect $\overline{B_{R_1}(o)}$, implying that $G_m \leq 0$ along γ_x, γ'. By the Gauss-Bonnet Theorem (Theorem 2.3.1), γ_x, γ' cannot intersect to form a bigon. Indeed, if such a bigon B existed with angles θ_1, θ_2, we have must have

$$0 \geq \int_B G_m = \theta_1 + \theta_2,$$

which is impossible. This implies that q cannot be conjugate to x along γ_x, a contradiction.

Now we consider the case where $M_m \setminus \overline{B_{R_1}(o)}$ is von Mangoldt. By Lemmas 2.5.15, 2.5.19, and 2.5.22, we can find a normal cut point y in C_q arbitrarily close to x such that $d(q,x) < d(q,y)$ and $\theta_x < \theta_y < \pi$. By Remark 2.5.21, there exists a minimal geodesic β_y connecting q to y such that

$$\measuredangle(\dot{\beta}_y(0), \dot{\tau}_q(0)) < \measuredangle(\dot{\gamma}_x(0), \dot{\tau}_q(0)),$$

and since y can be made arbitrarily close to x, we can ensure that β_y does not intersect $\overline{B_{R_1}(o)}$.

We now show that

$$\ell(\gamma_x) < \ell(\beta_y) \text{ and } r_{\gamma_x(s)} > r_{\beta_y(s)}$$

for all $s \in (0, \ell(\gamma_x))$. For each $s \in (0, \ell(\gamma_x))$, since $\theta_y > \theta_x$, there exists a unique value $t(s)$ of β_y giving us

$$\theta_{\alpha(s)} = \theta_{\beta_y(t(s))}.$$

Since γ_x, β_y cannot intersect in their interiors we have $r_{\beta_y(t(s))} < r_{\gamma_x(s)}$. Hence for any given s, the set

$$S_s := \{t \in (0, \ell(\beta_y)) \ \mid \ r_{\beta_y(t)} < r_{\gamma_x(s)}\}$$

is nonempty. Now fix $s_0 \in (0, \ell(\gamma_x))$. Let (a, b) be the connected component of S_{s_0} containing $t(s_0)$. If we show that $s_0 \in (a, b)$, then we will have $r_{\gamma_x(s_0)} > r_{\beta_y(s_0)}$. If $(0, \ell(\gamma_x)) \subseteq (a, b)$ then $s_0 \in (a, b)$ and there is nothing to prove, so we can assume $a > 0$ or $b < \ell(\gamma_x)$. We have

$$r_{\gamma_x(s_0)} = r_{\beta_y(a)} = r_{\beta_y(b)}, \quad 0 \leq \theta_{\beta_y(a)} < \theta_{\gamma_x(s_0)} = \theta_{\beta_y(t(s_0))} < \theta_{\beta_y(b)} < \pi$$

so the conditions for Lemma 2.5.14 are satisfied. It follows that

$$a = d(q, \beta_y(a)) < s_0 = d(q, \gamma_x(s_0)) < d(q, \beta_y(b)) = b,$$

implying $s_0 \in (a, b)$ and therefore $r_{\beta_y(s_0)} < r_{\gamma_x(s_0)}$. Since s_0 was arbitrary and $M_m \setminus \overline{B_{R_1}(o)}$ is von Mangoldt, we have $G_m(r_{\gamma_x(s)}) \leq G_m(r_{\beta_y(s)})$ for all $s \in [0, \ell(\gamma_x)]$. Recalling that q is conjugate to x along γ_x and applying the Sturm Comparison Theorem (Theorem 2.5.3), we have that q is conjugate to $\beta_y(t)$ along β_y for some $t \in (0, \ell(\gamma_x)]$. But this is impossible, since β_y minimizes the distance from q to y and $\ell(\beta_y) > \ell(\gamma_x)$. Hence q cannot have a cut point along γ_x, and this completes our proof.

□

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