The space of complete nonnegatively curved metrics on the plane

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In the last decade there has been considerable progress in studying spaces of Riemannian metrics that satisfy various curvature assumptions. In this report we are interested in the set $\mathcal{R}(N)$ of complete metrics of nonnegative sectional curvature on a fixed open connected manifold N. Here $\mathcal{R}(N)$ is given the topology of C^∞-uniform convergence on compact subsets, and more generally, this topology is given to all function spaces discussed below. Let $\mathcal{M}(N)$ denote the associated moduli space, i.e. the quotient space of $\mathcal{R}(N)$ by the pullback $\mathrm{Diff}(N)$-action.

Recall that any open complete manifold of nonnegative sectional curvature is diffeomorphic to a normal bundle of a compact totally convex submanifold called a soul. A soul is not unique but all souls of a given metric are isometric. Thus the isometry class of the soul is a basic invariant of the metric.

Kapovitch-Petrunin-Tuschmann [3] proved that if the normal bundle to a soul of some metric in $\mathcal{R}(N)$ has nonzero Euler class, then the diffeomorphism type of the soul defines a locally constant function on $\mathcal{R}(N)$ and $\mathcal{M}(N)$. More recently Belegradek-Kwasik-Schultz [1] showed that the result still holds when the "diffeomorphism type" of the soul is replaced by its "ambient isotopy type". These results lead to examples of manifolds for which $\mathcal{M}(N)$ has infinitely many path-components [3, 1, 2, 4].

If N admits a metric with a codimension one soul, then the topology of $\mathcal{M}(N)$ can be easily described in terms of the topology of the corresponding moduli spaces of its souls, of which there could be more than one [1].

The simplest case in which the methods of [3, 1] fail is when N has a codimension two soul with trivial normal bundle. To study the spaces of metrics for such manifolds it seems necessary to understand what happens for $N = \mathbb{R}^2$. It is easy to see that $\mathcal{R}(\mathbb{R}^2)$ is path-connected, and more generally, the following is true, which is the main result of this report.

Theorem 1. *Any countable (or finite) subset of $\mathcal{R}(\mathbb{R}^2)$ has the path-connected complement. The same holds for $\mathcal{M}(\mathbb{R}^2)$ in place of $\mathcal{R}(\mathbb{R}^2)$.*

The proof is based on the uniformization theorem, properties of subharmonic functions, and infinite-dimensional topology. The starting point is a classical result of Huber that any complete metric g on $\mathbb{R}^2$ of nonnegative curvature is conformal to the standard flat metric g_0. Thus g can be written as $\phi^*(e^{-2u}g_0)$ where u is a smooth function on $\mathbb{R}^2$, and ϕ is a self-diffeomorphism of $\mathbb{R}^2$. Nonnegativity of the curvature is equivalent to subharmonicity of u. Deciding which subharmonic functions give rise to complete metrics is more subtle, and is crucial for the proof. One can normalize ϕ so that it fixes two points of $\mathbb{R}^2$, say the complex numbers 0 and 1, so the map $(u, \phi) \to \phi^*(e^{-2u}g_0)$ defines a continuous bijection $C \times \mathrm{Diff}_{0,1}(\mathbb{R}^2) \to \mathcal{R}(\mathbb{R}^2)$, where C is a certain star-shaped set of subharmonic functions in the Fréchet space of smooth functions on $\mathbb{R}^2$, and $\mathrm{Diff}_{0,1}(\mathbb{R}^2)$ is the subgroup of $\mathrm{Diff}(\mathbb{R}^2)$ that fixes 0 and 1. There is also a continuous surjection $C \to \mathcal{M}(\mathbb{R}^2)$ whose fibers are closed subgroups of $\mathrm{Aff}(\mathbb{R}^2)$, the group of conformal automorphisms of $(\mathbb{R}^2, g_0)$. The topological group $\mathrm{Diff}_{0,1}(\mathbb{R}^2)$ is homeomorphic to the separable Hilbert space l_2. Also we shall make use of the classical result of infinite dimensional topology that the complement of any countable union of compact subsets of a separable Fréchet space is homeomorphic to l_2. Unfortunately, the homeomorphism type of C is unclear (to the author). There is a well-known topological classifications of closed convex subsets of separable Fréchet spaces, e.g. such a subset is homeomorphic to l_2 if and only if it is not locally compact. This classification does not seem to apply to C because it is probably neither closed nor convex; nevertheless, combining the classfication with a more detailed description of C allows one to show that the complement in C of any countable union of compact sets is path-connected, which easily implies Theorem 1.

Similar techniques yield Theorem 1 for S^2 in place of $\mathbb{R}^2$, and in fact, even stronger results hold in the S^2 case, which will be discussed elsewhere.

References

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Polyhedral analogue of Frankel conjecture.

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(joint work with Misha Verbitsky)

In this talk we propose a conjecture that can be seen as a polyhedral analogue of the celebrated Frankel's conjecture in Kähler geometry proved by Mori [3] and Siu-Yau [6]. Frankel conjecture states that a Kähler manifold with positive bisectional curvature is biholomorphic to a complex projective space. We explain an approach to our conjecture based on the theory of polyhedral Kähler manifolds, developed in [4]. Before stating the conjecture we need to give some definitions.

Definition. A polyhedral manifold is a manifold that is glued from a collection of Euclidean simplexes by identifying their hyperfaces via isometry.

Example. The surface of a tetrahedron in $\mathbb{R}^3$ represents a two-sphere with flat metric that has four singularities, namely conical points.

The singularities of a polyhedral metric happen in real codimension 2 (at codimension two faces) and at generic points the singularity is locally isometric to a product of $\mathbb{R}^{d-2}$ with a two-dimensional cone. A polyhedral manifold is called *non-negatively curved* if the cone angle at each face of codimension two is at most