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SOME CURIOUS KLEINIAN GROUPS AND HYPERBOLIC 5-MANIFOLDS

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Abstract. We present a new family of discrete subgroups of $\mathbf{SO}(5,1)$ isomorphic to lattices in $\mathbf{SO}(3,1)$. In some of the examples the limit sets are wildly knotted 2-spheres. As an application we produce complete hyperbolic 5-manifolds that are nontrivial plane bundles over closed hyperbolic 3-manifolds and conformally flat 4-manifolds that are nontrivial circle bundles over closed hyperbolic 3-manifolds.

§1. Introduction

Hyperbolic manifolds are complete Riemannian manifolds of constant negative sectional curvature. Any hyperbolic k-manifold is a quotient of the hyperbolic space $\mathbf{H}^k$ by a discrete group of isometries that acts freely.

Let M be a closed orientable hyperbolic k-manifold. The composition of the holonomy representation $\pi_1(M) \to \mathbf{SO}_0(k,1) \cong \mathrm{Isom}_+(\mathbf{H}^k)$ and the inclusion $\mathbf{SO}_0(k,1) \hookrightarrow \mathbf{SO}_0(n,1)$ is an isomorphism of the group $\pi_1(M)$ onto a discrete subgroup Γ of $\mathbf{SO}_0(n,1)$. The group Γ is convex-cocompact (i.e. geometrically finite without parabolics). In fact Γ stabilizes a totally geodesic copy of $\mathbf{H}^k$ in $\mathbf{H}^n$ and the Γ-action on $\mathbf{H}^k$ is cocompact. The hyperbolic manifold $\mathbf{H}^n/\Gamma$ is diffeomorphic to $M \times \mathbb{R}^{n-k}$ and the limit set of Γ is an unknotted $(k-1)$-sphere $\partial_\infty \mathbf{H}^k$ in $\partial_\infty \mathbf{H}^n$.

The quasiconformal deformations of Γ are also convex-cocompact [JM]. In general, the space $CC^n(\pi_1(M))$ of faithful discrete representations of $\pi_1(M)$ with the convex-cocompact image is an open subset of the real-algebraic variety $\mathrm{Hom}(\pi_1(M), \mathbf{SO}_0(n,1))$. It is conjectured that any faithful discrete representation of $\pi_1(M)$ lies in the closure of $CC^n(\pi_1(M))$.

1.1. Theorem. *There exists a closed oriented hyperbolic 3-manifold M and a representation $\rho \in CC^5(\pi_1(M))$ such that*

(1) *the hyperbolic manifold $\mathbf{H}^5/\Gamma$ is the total space of a nontrivial plane bundle over M, so it is not homeomorphiic to $M \times \mathbb{R}^2$;*
(2) *the limit set $\Lambda(\Gamma)$ of the group $\Gamma = \rho(\pi_1(M))$ is an unknotted 2-sphere;*

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(3) *the conformally flat manifold* $\Omega(\Gamma)/\Gamma$ *is the total space of a nontrivial circle bundle over* M *where* $\Omega(\Gamma) = \partial_\infty \mathbf{H}^5 \setminus \Lambda(\Gamma)$.

According to Anderson [An], the total space of any vector bundle over a closed negatively curved manifold admits a complete Riemannian metric of sectional curvature K with $-a^2 < K < -1$. Theorem 1.1 represents some progress towards the following problem.

1.2. Problem. *Let* E *be the total space of a vector bundle over a closed manifold* M. *Under what conditions does* E *admit a complete hyperbolic metric? (or, more generally, a complete locally symmetric Riemannian metric of negative sectional curvature?)*

The author showed in [Be1] that, if M is a closed negatively curved manifold of dimension ≥ 3, then only finitely many vector bundles over M can admit complete locally symmetric Riemannian metrics of negative sectional curvature.

Kapovich [Ka2] proved that only finitely many plane bundles over a closed orientable surface M_g of genus g can be given complete hyperbolic metrics. On the other hand Luo [L] (cf. [GLT], [Ka1], [Kui1,2]) have constructed a complete hyperbolic metric on a plane bundle over M_g provided its Euler number e satisfies $|e| < g$. Hyperbolic structures on some nonorientable vector bundles over nonorientable surfaces have been constructed in [Be2].

1.3. Theorem. *There is a closed hyperbolic* 3*-manifold* M *and a representation* $\rho \in CC^5(\pi_1(M))$ *such that the limit set of the group* $\rho(\pi_1(M))$ *is a wild* 2*-sphere in* $\partial_\infty \mathbf{H}^5$.

First examples of this sort are due to Gromov, Lawson and Thurston [GLT] who found convex-cocompact surface groups actions on $\mathbf{H}^4$ such that the limit sets are wild circles in $\partial_\infty \mathbf{H}^4$. Apanasov and Tetenov [AT] have discovered a closed hyperbolic 3-manifold M and a representation $\rho \in CC^4(\pi_1(M))$ such that the limit set of the group $\rho(\pi_1(M))$ is a wild 2-sphere in $\partial_\infty \mathbf{H}^4$.

To prove 1.1 and 1.3 we build fundamental domains for the discrete groups in question. Roughly speaking, these fundamental domains are "suspensions" of the necklace-shaped fundamental domains of [GLT] and [Kui1,2]. This is similar to suspending an (un)knotted circle in S^3 to produce an (un)knotted 2-sphere in S^4. In the proof of 1.1 the necklaces are unknotted; knotted necklaces are used in 1.3.

1.4. Corollary. *There is a closed oriented hyperbolic* 3*-manifold* M *such that the space* $CC^5(\pi_1(M))$ *is not connected.*

In fact any two representations lying in the same connected component of $CC^n(\pi_1(M))$ are topologically conjugate on $\mathbf{H}^n \cup \partial_\infty \mathbf{H}^n$. Therefore, 1.4 follows from either of the two theorems above.

Note: recently Apanasov [Ap1] announced some of the results of the present paper; he uses his block-building method to produce discrete groups.

§2. Fundamental polyhedrons

2.1. Hyperbolic and spherical polyhedrons

For the sake of visualization we always deal with spherical polyhedrons at infinity rather than hyperbolic polyhedrons. Recall that a (convex) hyperbolic polyhedron in $\mathbf{H}^n$ is the intersection of a collection of half spaces. Similarly, a spherical polyhedron in $\mathbf{S}^{n-1} = \partial_\infty \mathbf{H}^n$ is defined to be the intersection of a collection of round (conformal) balls.

The ideal boundary of a half space is a round (conformal) ball in the sphere at infinity. Conversely, given a round ball in $\partial_\infty \mathbf{H}^n$, its convex hull in $\mathbf{H}^n$ is a half space. Thus convex hyperbolic polyhedrons and spherical polyhedrons are in one-to-one correspondence.

Any polyhedron is the union of its faces; we assume faces are closed. Thus, the polyhedron itself is a face of codimension zero. Faces of codimension one (two, three) are called sides (edges, vertices, respectively).

2.2. Poincaré's Polyhedron Theorem

The way we prove discreteness in this paper is by applying Poincaré's Polyhedron Theorem. We refer to [EP] for a proof and definitions.

Basically Poincaré's Polyhedron Theorem says that given a polyhedron P in the hyperbolic n-space and a collection of isometries $g_1, \dots g_k \dots$ pairing the sides of P under certain conditions on these "initial data" the group Γ generated by $g_1, \dots g_k \dots$ is discrete and P is a fundamental polyhedron for Γ. The Γ-images of P tesselate $\mathbf{H}^n$. On the other hand, any discrete group of hyperbolic isometries arises this way because it has a fundamental polyhedron (e.g. a Dirichlet fundamental polyhedron).

However, in practice, the conditions of Poincaré's Theorem are extremely difficult to verify. We can handle the verification in 4.4 only because in our case the fundamental polyhedron and the side pairing transformations have certain symmetries.

Yet in the special case when P has finitely many sides and all the isometries $g_1, \dots, g_k$ are reflections in the sides of Φ most conditions of Poincaré's theorem are redundant. Namely we only have to check that all dihedral angles of Φ are submultiples of π. In this case we say that $\Gamma =< g_1, \dots, g_k >$ is a *reflection group*.

We now describe the conditions to be checked in Poincaré's Polyhedron Theorem (see [EP] for details). We start with a convex polyhedron P in $\mathbf{H}^n$. The condition *Pairing* says that side pairing transformations of P are defined so that each of them maps the interior of P to a subset disjoint from P. The condition *Finite* means that P has finitely many sides. *First Metric* is the condition that any two faces of P are in positive distance. The condition *Cyclic* guarantees that copies of P can be laid out one by one around any edge so that after finitely many steps P arrives at its initial position.

More generally, one can start with a collection $\{P_i : i \in I\}$ of convex polyhedrons. All the conditions above make sense. In addition, we have to assume the condition *Connected* that requires the identification space of $\bigcup_{i\in I} P_i$ induced by *Pairing* be connected.

2.3. Suspensions over necklaces

In this section we construct a spherical polyhedron in $\mathbf{S}^4 = \partial_\infty \mathbf{H}^5$. Under some extra assumptions the polyhedron will be a fundamental set for a discrete group.

Denote by $\mathbb{S}^3$ the unit sphere in $\mathbb{R}^4$. We identify $\overline{\mathbb{R}^4}$ with $\mathbf{S}^4$.

A *necklace* $N = N_{\alpha,\nu}$ is a finite cyclically ordered sequence of ν closed round balls in $\mathbf{S}^4$ satisfying the following conditions.

(i) The radii of the balls are equal.

(ii) Every ball in N meets the unit ball in $\mathbb{R}^4$ in an exterior angle $\pi/2$.

(iii) There is $\alpha \in (0, \pi/2]$ such that adjacent balls meet in an exterior angle α.

We say that a necklace is *embedded* (cf. [Kui2]), if nonadjacent balls in N do not meet. Fix $\beta \in (0, \pi/2)$ and a necklace N containing ν balls. Consider the (uniquely determined) round balls B_0 and B_∞ in $\overline{\mathbb{R}^4}$ with centers at 0 and ∞, respectively, meeting the balls of N in an exterior angle β. The inversion i in the unit sphere $\mathbf{S}^3$ transposes B_0 and B_∞. We shall refer to these two balls as the *big balls* (or just the *b-balls*).

Now we consider an arbitrary pair of adjacent balls B and B' centered at points C and C', respectively. Clearly, the balls B, B' and B_0 cover the triangle $\triangle C0C'$ if and only if $\alpha + 2\beta \geq \pi$.

Assume that $\alpha + 2\beta < \pi$ (see Figure on the next page). Denote by X and Y the points where the bisectrix of the angle $\angle C0C'$ meets $B \cap B'$ and ∂B_0, respectively. It is easy to see that, there exists a unique round ball B_T centered at a point $T \in (X, Y)$ such that B_T meets each of the balls B, B' and B_0 in an exterior angle $\pi/2$. It is clear that the balls B_T, B_0, B and B' cover $\triangle C0C'$; for this reason, we say that the ball B_T fills the hole between the balls B_0, B and B'.

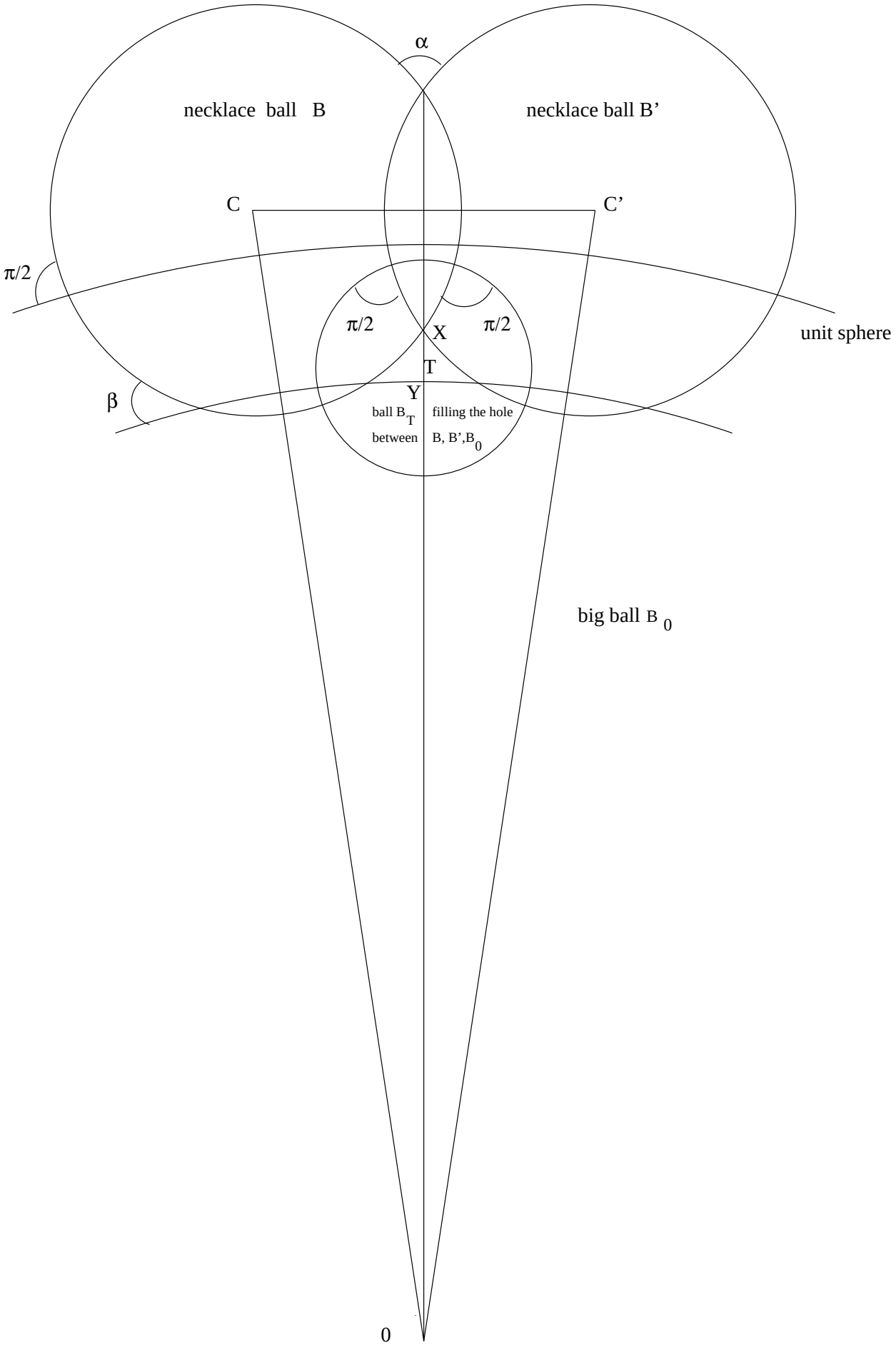
Clearly, the ball $i(B_T)$ is orthogonal to B_∞, B and B', and the balls $i(B_T)$, B_∞, B and B' cover the triangle $\triangle C\infty C'$.

Varying the pairs of adjacent balls we get ν balls of the form B_T, and then ν other balls of the form $i(B_T)$. We shall refer to these 2ν balls filling holes as the *f-balls*.

Under the assumption $\alpha + 2\beta < \pi$, we call the set $S_\beta N$ of f-balls, the b-balls and the necklace balls the *suspension over the necklace* N. Otherwise we use this notation for the set of b-balls and necklace balls. (We emphasize that the suspension is uniquely determined by N and β.)

We say that a suspension over the necklace N is *embedded* if the f-balls are pairwise disjoint and each f-ball meets exactly two necklace balls.

Given $S_\beta N$, denote by $F = F(N, \beta)$ the spherical polyhedron obtained by removing from $\mathbf{S}^4$ the internal points of the balls of the suspension. The


α
necklace ball B
necklace ball B'
C
C'
π/2
π/2
π/2
X
unit sphere
T
Y
β
ball B_T filling the hole
between B, B',B_0
big ball B_0
0

convex hull of $F = F(N, \beta)$ in $\mathbf{H}^n$ is a convex finitely-sided polyhedron.

2.4. The template construction

We describe some preliminary construction introduced by Gromov, Lawson, Thurston [GLT] (see also [Kui1,2]). Fix $\varepsilon \in [0, \pi/2)$ and relatively prime integers q and p with $1 \le q < p$. Consider the curve $\Gamma_{q,p,\varepsilon}$ in the unit sphere

$$\mathbf{S}^3 = \{(z, w) \in \mathbb{C}^2 = \mathbb{R}^4 : |z|^2 + |w|^2 = 1\}$$

parametrized by $\Gamma(t) = (e^{qit}\cos\varepsilon,\, e^{pit}\sin\varepsilon), t \in [0, 2\pi]$.

For $\omega = e^{2\pi i/\nu}$ and $j \in \mathbb{Z}/\nu\mathbb{Z}$, the unit vectors $\eta_j = (\omega^{qj}\cos\varepsilon,\, \omega^{pj}\sin\varepsilon)$ subdivide $\Gamma_{q,p,\varepsilon}$ into equal parts. The polygonal curve $\gamma_{q,p,\varepsilon,\nu}$ in $\mathbf{S}^3$ whose vertices are η_j is the so called *template* (see [GLT]).

2.5. Templates and Suspensions

Here is a concrete example of a suspension. Given $\alpha \in (0, \pi/2]$ and $\nu \ge 5$, we consider a unique necklace $\mathfrak{N} = \mathfrak{N}_{\alpha,\nu,q,p,\varepsilon}$ such that for any j the intersection of jth ball of $\mathfrak{N}$ and $\mathbf{S}^3$ is a metric ball centered at η_j. For $\beta \in (0, \pi/2)$, we next consider the suspension $S_\beta\mathfrak{N}$. Note that $\mathfrak{N}$ and $S_\beta\mathfrak{N}$ are uniquely determined by the choice of $q, p, \varepsilon, \nu, \alpha$ and β.

2.6. Example. Suppose that $\varepsilon = 0$ and $q = 1$. Then the template $\gamma_{q,p,\varepsilon,\nu}$ is a plane regular ν-gon. It is straightforward to verify in this case that if $\nu \ge 5$, the corresponding necklace $\mathfrak{N}_0 = \mathfrak{N}_{\alpha,\nu,1,p,0}$ and the suspension $S_\beta\mathfrak{N}_0$ are embedded.

2.7. Claim. *For ν large enough, the necklace $\mathfrak{N} = \mathfrak{N}_{\alpha,\nu,q,p,\varepsilon}$ and the suspension $S_\beta\mathfrak{N}$ are embedded.*

Proof. For fixed q, p and ε, we vary ν. As ν tends to ∞, the template $\gamma_{q,p,\varepsilon,\nu}$ approximates the smooth knot $\Gamma_{q,p,\varepsilon}$. Thus, locally, the necklace $\mathfrak{N}$ looks like the necklace $\mathfrak{N}_0$ whose balls centered at the vertices of a plane regular ν-gon. Since, for $\nu \ge 5$, $\mathfrak{N}_0$ and $S_\beta\mathfrak{N}_0$ are embedded so are $\mathfrak{N}$ the $S_\beta\mathfrak{N}$, for every ν large enough. □

§3. Reflection groups

3.1. Convention. Throughout the section we suppose that

$$\alpha = \frac{\pi}{l} \text{ where } l \in \mathbb{Z},\ l \ge 4 \quad \text{and} \quad \beta = \frac{\pi}{m} \text{ where } m \in \mathbb{Z},\ m \ge 3.$$

3.2. Reflection groups coming from suspensions

Given such α, β and an integer $\nu \ge 5$, we fix a necklace N and the suspension $S_\beta N$. Consider the group $\Gamma = \Gamma(N, \beta)$ generated by the inversions in the spheres that bound the balls of the suspension.

Suppose that the necklace and the suspension considered are embedded. Then by Poincaré's Polyhedron Theorem [EP] applied to the natural actions of Γ in $\mathbf{H}^5$, the group Γ is discrete and F is a spherical fundamental polyhedron for Γ.

Poincaré's theorem allows us to write down a presentation of Γ. Namely, Γ is generated by reflections in the sides of F (the number of the sides is $3\nu+2$ when $\alpha+2\beta<\pi$ and $\nu+2$ otherwise). Each generating reflection i gives the relation $i^2=1$. If the angle at an edge e is equal to π/n, the corresponding relation is $(i_1i_2)^n=1$, where i_1 and i_2 are the reflections in the sides of F abutting e. The relations above define Γ. Note that the integers ν, l and m determine the presentation uniquely.

3.3. Example: reflection lattices in Isom($\mathbf{H}^3$). Assume $N=\mathfrak{N}_0$, the necklace whose balls are centered at the vertices of a plane regular ν-gon. Since $\nu\geq 5$, the necklace $\mathfrak{N}_0$ and the suspension $S_\beta\mathfrak{N}_0$ are embedded (see 2.6). Then the group $\Gamma=\Gamma(\mathfrak{N}_0,\beta)$ is the natural extension to $\mathbf{S}^4$ of a discrete cocompact subgroup of the group of isometries of $\mathbf{H}^3$. In particular, the limit set of such a Γ is a round 2-sphere.

3.4. Theorem. *Given $k\in\{1,2\}$, let Γ_k be a reflection group (i.e. Γ is a discrete group generated by reflections in the sides of a convex finitely-sided polyhedron P_k). Then the actions of Γ_1 and Γ_2 on $\mathbf{H}^n$ are topologically conjugate if amd only if there is a homeomorphism of P_1 onto P_2 preserving the face structure.*

Proof. Suppose that there is a homeomorphism $f: P_1\to P_2$ preserving the face structure. Then f defines an isomorphism $\phi_f:\Gamma_1\to\Gamma_2$. Clearly, f is ϕ_f-equivariant. Hence, f can be extended to a ϕ_f-equivariant self-homeomorphism of $\mathbf{H}^n$.

Conversely, assume $\tilde f$ is a homeomorphism of $\mathbf{H}^n$ equivariant with respect to an isomorphism $\phi_f:\Gamma_1\to\Gamma_2$. For any reflection group Γ, the union of all the hyperplanes which are the fixed-point-sets of reflections in Γ splits $\mathbf{H}^n$ into the disjoint union of convex polyhedrons; these are the so-called chambers of Γ. For example, the chambers of Γ_k are of the form $\gamma(P_k)$ where $\gamma\in\Gamma_k$. Given a hyperplane H_r that is the fixed-point-set of a reflection $r\in\Gamma_1$, the set $\tilde f(H_r)$ is the fixed-point-set of a reflection $\phi_f(r)\in\Gamma_2$; so $\tilde f(H_r)$ is a hyperplane. It follows that $\tilde f$ takes chambers of Γ_1 to chambers of Γ_2 preserving the face structure. In particular, $\tilde f(P_1)$ is a chamber, therefore $\tilde f(P_1)=\gamma(P_2)$ for some $\gamma\in\Gamma_k$. Hence $\gamma^{-1}\circ\tilde f: P_1\to P_2$ is the desired homeomorphism. $\square$

3.5 Corollary. *There is a cocompact reflection subgroup $\Gamma\leq Isom(\mathbf{H}^3)$ such that $CC^5(\Gamma)$ is not connected.*

Proof. Any embedded necklace N defines an isotopy class of knots in the unit sphere $\mathbf{S}^3\subset\mathbb{R}^4$. Given the suspension $S_\beta N$, let F be the spherical polyhedron obtained by removing from $\mathbf{S}^4$ the internal points of the balls of

the suspension. The interior of F is homeomorphic to $C \times \mathbb{R}$ where C is the complement of a knot in the isotopy class defined by N. Thus $\pi_(F) \cong \pi_1(C)$.

For α and ν as in 3.1 and consider embedded necklaces $\mathfrak{N} = \mathfrak{N}_{\alpha,\nu,q,p,\varepsilon}$ and $\mathfrak{N}_0 = \mathfrak{N}_{\alpha,\nu,1,p,0}$ (see 2.5-2.7). We assume $\varepsilon \in (0, \pi/2)$ and $p > q > 1$; notice that in this case the necklace $\mathfrak{N}_{\alpha,\nu,q,p,\varepsilon}$ is "knotted". Reflections in the sides of the corresponding spherical polyhedrons $F = F(\mathfrak{N}, \beta)$ and $F_0 = F(\mathfrak{N}_0, \beta)$ generate discrete groups Γ and Γ_0 (see 3.2). Note that $\Gamma \cong \Gamma_0$.

Assume $CC^5(\Gamma)$ is connected. Then according to Theorem 8.1 the groups Γ and Γ_0 are topologically conjugate on $\mathbf{H}^5$. Therefore, by Theorem 3.4 the chambers F and F_0 are homeomorphic. The group of a nontrivial (tame) knot cannot be cyclic, so $\pi_1(F)$ is not isomorphic to $\mathbb{Z}$. On the other hand $\pi_1(F_0) \cong \mathbb{Z}$, a contradiction. $\square$

§4. Main construction

In §3 we have constructed some discrete reflection subgroups of $\mathrm{Isom}(\mathbf{H}^5)$ that are isomorphic to cocompact lattices in $\mathrm{Isom}(\mathbf{H}^3)$. The groups will be used in the proof of Theorem 1.3. Now we construct more subtle examples of discrete subgroups of $\mathrm{Isom}(\mathbf{H}^5)$ isomorphic to cocompact lattices in $\mathrm{Isom}(\mathbf{H}^3)$. These are to be used in the proof of Theorem 1.1.

4.1. Convention. Throughout the section we suppose that

$$\alpha = \frac{2\pi}{\nu} \text{ where } \nu \in \mathbb{Z},\ \nu \geq 5 \quad \text{and} \quad \beta = \frac{\pi}{2m} \text{ where } m \in \mathbb{Z},\ m \geq 2.$$

4.2. Set up

Given such α and β, we fix a necklace N that contains exactly ν balls and the suspension $S_\beta N$. Assume that N and $S_\beta N$ are embedded. Notice that the suspension contains f-balls since $\alpha + 2\beta < \pi$.

We are going to define side pairing transformations for the spherical polyhedron $F = F(N, \beta)$. First, we introduce some notations.

Each of the balls of the suspension determines the only side of F (that is the intersection of the ball and F). For this reason, we use the following terms: *b-sides, necklace sides, f-sides.*

It follows from our construction that the inversion i in the unit sphere in $\mathbb{R}^4$ transposes b-sides, preserves necklace sides and carries f-sides to f-sides. Given a side s of F, denote by i_s the reflection in the side. Now let s be an arbitrary necklace side determined by a necklace ball B_j. Let B_{j-1} and B_{j+1} be the necklace balls adjacent to B_j. Consider the segment in $\mathbb{R}^4$ connecting the centers of B_{j-1} and B_{j+1} and the 3-plane p in $\mathbb{R}^4$ meeting the segment in the middle and orthogonal to it. It follows from our construction that p passes through the origin and the center of B_j; moreover, the ball filling the hole between B_{j-1}, B_j and the ball filling the hole between B_j, B_{j+1} are symmetric with respect to p. Hence the reflection r_p in the plane p preserves the side s.

We define side pairing transformations for F as follows. If s is a necklace side, the side pairing transformation of s is defined to be the half-turn $r_p \circ i_s$ that preserves s. Otherwise, the side pairing transformation of the sides s and $i(s)$ is defined to be $i \circ i_s$. Denote by Γ the group generated by all the transformations pairing sides of F.

Below we describe conditions needed to make Γ discrete.

4.3. Torsion Condition

Let $\mathbb{B}^4$ be the unit ball in $\mathbb{R}^4$ having hyperbolic metric; this is a standard model for the hyperbolic 4-space. Denote by H the subgroup of Γ generated by the half-turns pairing necklace sides. Clearly, H acts on $\mathbb{B}^4$ by hyperbolic isometries. The half-turns generating H pair the sides of the hyperbolic polyhedron P_H obtained by removing from $\mathbb{B}^4$ the interior points of the necklace balls.

In fact, the group H has been studied in [Kui1,2]. All the edges of P_H belong to one cycle and the sum of the angles at these edges is equal to 2π. Denote by h a cyclic transformation of an edge e of P_H. Kuiper [Kui1] has observed that h has a fixed point in e which is precisely the orthogonal projection of 0 into the 2-plane e. Let $e^\perp$ be a 2-plane in $\mathbb{B}^4$ meeting e in the fixed point and orthogonal to e. Then h is the product of two rotations, ρ and $\rho^\perp$, whose fixed point sets are e and $e^\perp$, respectively. Since the sum of the angles at the edges of P_H is equal to 2π the rotation $\rho = 1$.

Kuiper called the rotation $\rho^\perp$ the *torsion of* P_H. He has shown that the condition $\rho^\perp = 1$ (the so-called *Torsion Condition*) is sufficient to make H discrete.

4.4. Lemma. *Let* Γ *be the group constructed in* 4.2 *for an embedded suspension over an embedded necklace. If the Torsion Condition holds, then* Γ *is discrete and* F *is its spherical fundamental polyhedron.*

Proof. Let P be the convex hull of F in $\mathbf{H}^5$. Since any side of P is the convex hull of a unique side of F, we use the terms *f-sides, b-sides, necklace sides* for sides of P. Also we use the same symbols for inversions in $\overline{\mathbb{R}^4}$ and their extensions to $\mathbf{H}^5$. We are to apply Poincaré's Polyhedron Theorem [EP, Theorem 4.14] to show that P is a hyperbolic fundamental polyhedron for Γ.

The conditions *Connected* and *Finite* are satisfied trivially. The condition *First Metric* holds since no two balls of the suspension are tangent. Clearly the condition *Pairing* can be reduced to (i).

(i) For any side pairing transformation g, the sets $g(\mathrm{int}(P))$ and $\mathrm{int}(P)$ are disjoint.

To verify (i) fix an arbitrary side pairing transformation g that pairs the sides s and $g(s)$. Then either $g = i \circ i_s$ or $g = r_p \circ i_s$. First, assume $g = i \circ i_s$. Since P is convex, $i_s(\mathrm{int}(P))$ and $\mathrm{int}(P)$ are disjoint. Therefore, $(i \circ i_s)(\mathrm{int}(P))$ and $i(\mathrm{int}(P)) = \mathrm{int}(P)$ are disjoint as desired. Second, suppose that $g = r_p \circ i_s$. Consider a (unique) half space H with $H \cap P = s$. Then

$i_s(\mathrm{int}(P)) \subset H$, becasuse P is convex. By construction, H is r_p-invariant. Thus, $(r_p \circ i_s)(\mathrm{int}(P))$ and $\mathrm{int}(P)$ are disjoint and hence (i) holds.

Since *Finite* implies *First Cyclic* [EP, p128], the condition *Cyclic* is reduced to (ii) and (iii).

(ii) For each edge cycle, the sum Σ of the angles at the edges of the cycle is of the form $2\pi/n$, where n is a positive integer.

(iii) Let e be an edge of P from an edge cycle with $\Sigma = 2\pi/n$ and let h be a cycle transformation of e. Then $h^n = 1$.

We now check (ii) and (iii). There are four types of edge cycles; we shall consider them separately.

Start with an edge cycle consisting of two edges with the right angles at them; there are exactly ν such cycles. Each of the edges lies on the boundary of a b-side and a f-side. Fix an edge e in the cycle and denote by b and f the b-side and f-side, respectively, abutting e. Then for the cycle transformation h of e we have

$$h = (i_f \circ i) \circ (i \circ i_b) = i_f \circ i_b.$$

Hence h is the rotation about e by the angle π. So, in the case considered, $h^2 = 1$ and $\Sigma = 2 \cdot \pi/2 = \pi$. Thus, in this case (ii) and (iii) have been checked.

Besides the cycles already considered, there are ν other cycles consisting of two edges; they can be described as follows. Consider an arbitrary necklace side s and the b-sides b and $i(b)$. Let an edge e lie on the boundary of s and b. Then the edge $i(e)$ lies on the boundary of s and $i(b)$. We describe a trip of e round the edge cycle. The transformation $i \circ i_b$ maps e to $i(e)$. Then $r_p \circ i_s$ carries $i(e)$ to itself. Then $i_b \circ i$ returns $i(e)$ back to e and, finally, $i_s \circ r_p$ carries e to itself. So

$$h = (i_s \circ r_p) \circ (i_b \circ i) \circ (r_p \circ i_s) \circ (i \circ i_b), \qquad \Sigma = 4 \cdot \pi/2m = 2\pi/m.$$

Since the inversions in orthogonal spheres always commute, r_p commutes with i_s, i, i_b, and i commutes with i_s. Then $h = i_s \circ i_b \circ i_s \circ i_b$. Clearly, $i_s \circ i_b$ is a rotation about e by the angle π/m and therefore $h^m = 1$. So (i) and (ii) hold.

There are exactly ν cycles consisting of four edges; they can be described as follows. Consider an arbitrary necklace side s and the corresponding 3-plane p; put $r = r_p$. Also consider those f-sides which meet s; if we denote one of them by f, they are f, $r(f)$, $i(f)$ and $r(i(f))$. Let an edge e lie on the boundary of s and f. We describe a trip of e round the edge cycle. The transformation $i \circ i_f$ maps e to $i(e)$. Then $r \circ i_s$ carries $i(e)$ to $r(i(e))$. Then $i_{r(f)} \circ i$ maps $r(i(e))$ to $r(s)$ and finally, $i_s \circ r$ returns $r(e)$ back to e. So

$$h = (i_s \circ r) \circ (i_{r(f)} \circ i) \circ (r \circ i_s) \circ (i \circ i_f), \qquad \Sigma = 4 \cdot \pi/2 = 2\pi.$$

As above, the inversion i_s commutes with i, r, $i_{r(f)}$ and the inversion i commutes with r. Then $h = r \circ i_{r(f)} \circ r \circ i_f$. Since the sides f and $r(f)$ are symmetric with respect to p we get $h = 1$. Hence (i) and (ii) hold.

It remains to consider the only edge cycle consisting of ν edges. Each of the edges lies on the boundaries of two necklace sides. Applying half-turns we arrange a travel of a chosen edge e round the cycle. According to 4.5, $h = \rho \circ \rho^{\perp}$ is a cycle transformation of e. Since $\Sigma = \nu \cdot 2\pi/\nu = 2\pi$ we have $\rho = 1$. In addition, by the *Torsion Condition*, $\rho^{\perp} = 1$. So $h = 1$; therefore the conditions (i) and (ii) hold.

The proof of the lemma is complete. □

4.5. Example: cocompact lattices in Isom($\mathbf{H}^3$). Assume $N = \mathfrak{N}_0$, the necklace whose balls are centered at the vertices of a plane regular ν-gon (see 2.6). Since $\nu \geq 5$, the necklace $\mathfrak{N}_0$ and the suspension $S_\beta\mathfrak{N}_0$ are embedded (see 2.6). The *Torsion Condition* holds trivially. Then the group $\Gamma = \Gamma(\mathfrak{N}_0, \beta)$ constructed in 4.2 is the natural extension to $\mathbf{S}^4$ of a discrete cocompact subgroup of the group of isometries of $\mathbf{H}^3$. In particular, the limit set of such a Γ is a round 2-sphere.

We next describe a general construction for which all assumptions of 4.4 are satisfied. Start with the necklace $\mathfrak{N} = \mathfrak{N}_{\alpha,\nu,q,p,\varepsilon}$ and the suspension $S_\beta N$. keeping in mind that, by 4.1, $\alpha = 2\pi/\nu$ and $\beta = \pi/2m$.

4.6. Proposition. *Given q, p and m, there exist $\varepsilon \in (0, \pi/2)$ and ν such that the necklace $\mathfrak{N} = \mathfrak{N}_{\alpha,\nu,q,p,\varepsilon}$ and the suspension $S_\beta\mathfrak{N}$ are embedded and the Torsion Condition holds.*

Proof. Kuiper [Kui2] has proved that his Torsion Condition is equivalent to the following condition discovered by Gromov, Lawson and Thurston [GLT]

$$T = \frac{\nu\tau}{2\pi} \in \mathbb{Z}, \tag{1}$$

where τ is the torsion along each edge of the template $\gamma_{q,p,\varepsilon,\nu}$ (see [GLT]) and $2\pi T$ is the total torsion of the template. The calculations below were done in [GLT] and [Kui1]:

$$\begin{aligned}\cos\tau &= \frac{\cos^2\varepsilon\cdot\sin^2(2q\pi/\nu)\cdot\cos(2p\pi/\nu) + \sin^2\varepsilon\cdot\sin^2(2p\pi/\nu)\cdot\cos(2q\pi/\nu)}{\cos^2\varepsilon\cdot\sin^2(2q\pi/\nu) + \sin^2\varepsilon\cdot\sin^2(2p\pi/\nu)} \\ &= \frac{\sin^2 2q\omega\cdot\cos 2p\omega + u\cdot\sin^2 2p\omega\cdot\cos 2q\omega}{\sin^2 2q\omega + u\cdot\sin^2 2p\omega};\end{aligned} \tag{2}$$

here $\omega = \pi/\nu$ and $u = \mathrm{tg}^2\,\varepsilon$.

Taking into account that $0 < q < p$ and assuming $\nu > 2p$ we get $0 < 2\omega q < 2\omega p < \pi$. So $\cos 2\omega q > \cos 2\omega p$. Then (2) implies $\cos 2\omega q > \cos\tau > \cos 2\omega p$.

Since $\tau \in (0, \pi)$ [GLT], we get $q < T = \nu\tau/2\pi < p$. It follows from (2) that

$$\mathrm{tg}^2\,\varepsilon = u = \frac{\sin^2 2q\omega\cdot\sin(p-T)\omega\cdot\sin(p+T)\omega}{\sin^2 2p\omega\cdot\sin(T-q)\omega\cdot\sin(T+q)\omega}. \tag{3}$$

Suppose that T is an integer. Then, for any ε satisfying (3) and for any $\nu > 2p$, the template $\gamma_{q,p,\varepsilon,\nu}$ is such that the Torsion condition (1) holds. For fixed q, p and T, we vary ν. As ν tends to ∞, $u = u(\nu)$ tends to

$$u_\infty = \frac{q^2}{p^2} \cdot \frac{p^2 - T^2}{T^2 - q^2}.$$

Since $u_\infty > 0$ we have $u_\infty = \mathrm{tg}^2\varepsilon_\infty$, for some $\varepsilon_\infty \in (0, \pi/2)$. Therefore, as ν tends to ∞, the template $\gamma_{q,p,\varepsilon(\nu),\nu}$ approaches the *smooth* knot $\Gamma_{q,p,\varepsilon_\infty}$. Hence, locally, the necklace $\mathfrak{N}_{\alpha,\nu,q,p,\varepsilon(\nu)}$ looks like the necklace $\mathfrak{N}_0 = \mathfrak{N}_{\alpha,\nu,q,p,0}$ whose balls centered at the vertices of a plane regular ν-gon. Since $\mathfrak{N}_0$ and $S_\beta\mathfrak{N}_0$ are embedded, so are $\mathfrak{N}_{\alpha,\nu,q,p,\varepsilon(\nu)}$ and $S_\beta\mathfrak{N}_{\alpha,\nu,q,p,\varepsilon(\nu)}$, for every ν large enough. □

4.7. Remark. Let us discuss the effect of the choices of α and β and ν we made in 3.1 and 4.1. Any point of ∂F lies in the intersection of at most three sides. Moreover, given a collection of sides of F, the intersection of the sides is not a point; it is either empty or it has a positive dimension.

4.8. Theorem. *We assume that* Γ *is either the group satisfying the assumption of* 4.4 *or the group constructed in* 3.2. *Then* Γ *is convex-cocompact, in particular the limit set of* Γ *is homeomorphic to the* 2*-sphere.*

Proof. Since P has finitely many sides, Γ is geometrically finite [Bo]. Hence, it suffices to prove that Γ has no parabolics. The limit set of Γ consists of conical limit point and bounded parabolic points [Bo]. According to [Tu, Theorem 2.4], each bounded parabolic point is Γ-equivalent to a point that lies on $F = \partial_\infty P$ and, conversely, any limit point that lies on F is a bounded parabolic point; moreover, F contains at most finitely many limit points.

Assume $p \in F$ is a (bounded parabolic) limit point. So the stabilizer Γ_p of the point p in Γ is an infinite group. In particular, p cannot lie in the interior of F, since no point of the interior of F is stabilized by an element of Γ. Thus $p \in \partial F$. The intersection of F and the Γ-orbit of p is a finite set $\{\gamma_k(p) : k = 0, \dots, n\}$ where γ_0 is the identity.

Let B_p be a Γ_p-invariant horoball centered at p. Since Γ is geometrically finite, one can choose B_p so small that

(i) $B_p \cap \gamma(B_p) = \emptyset$ for all $\gamma \in \Gamma$ and,

(ii) if $P \cap \gamma(B_p) \neq \emptyset$, then $\gamma = \gamma_k$ for some $k = 0, \dots, n$ and,

(iii) for any $k = 0, \dots, n$, the horoball $\gamma_k(B_p)$ intersects only those faces of P whose ideal boundaries contain $\gamma_k(p)$.

This follows from [Bo, Proposition 4.4] and the observation that any standard parabolic region at p contains a horoball centered at p.

The horosphere $S_p = \partial B_p$ is identified with $\mathbb{R}^4$ and Γ_p acts on S_p by euclidean isometries (to visualize it, one can conjugate Γ in $\mathrm{Isom}(\mathbf{H}^5)$ so that $p = \infty \in \mathbf{H}^5$).

The fundamental polyhedron for this action is the union of convex (closed) euclidean polyhedrons $E_k = \gamma_k^{-1}(P \cap \gamma_k(S_p))$; note that $E_0 = P \cap S_p$. It

defines a tesselation of S_p where any tile is of the form $\gamma(E_k)$ with $\gamma \in \Gamma_p$ and $k = 0, \dots, n$.

We are to show that the tesselation has finitely many tiles (therefore, Γ_p is a finite group which contradicts the assumption that p is a bounded parabolic point).

All tiles have the same combinatorial structure. In fact, according to 4.7, there are only three possibilities as follows. If p belongs to exactly one side, then E_k is a half space. In case p lies on exactly two sides, then E_k is the (nonempty) intersection of two half spaces. Finally, if p belongs to three sides, E_k is the intersection of three half spaces such that their boundary hyperplanes have a common face.

We conclude that any tile must have exactly one face of the smallest dimension. This face has the property that any euclidean ray that starts at a point of the face and passes through a point of the tile is contained in the tile.

Fix a tile E with the face f of the smallest dimension. Choose a point $q \in f$. A small neighborhood of q in S_p is covered by finitely many tiles abutting f. Since all tiles have the same combinatorial structure, f is the face of the smallest dimension in these tiles. Hence the union of the tiles contains any ray that starts at q, hence the tiles cover S_p. Thus S_p is covered by finitely many tiles and, hence, Γ_p is a finite group. A contradiction with the choice of p. Therefore Γ is convex-cocompact as wanted.

Finally, since Γ is a convex-cocompact group isomorphic to a cocompact lattice in $\mathrm{Isom}(\mathbf{H}^3)$, the limit set of Γ is homeomorphic to the 2-sphere [Tu]. □

4.9. Boundary groups

A slight modification of our construction yields discrete groups on the boundary of $CC^n(\Gamma)$ where Γ is a cocompact reflection subgroup of the group of isometries of the hyperbolic 3-space. Such groups are known in abundance in the classical Kleinian group theory. For example, if Γ is a surface group, the boundary of $CC^3(\Gamma)$ contains regular b-groups (being geometrically finite group with parabolics) as well as degenerate groups. However, if Γ is a lattice in a higher dimensional hyperbolic space, little is known. First examples of boundary groups are due to Apanasov [Ap2]. We provide a very simple example of a boundary group.

4.10. Theorem. *There is a cocompact reflection subgroup $\Gamma \leq Isom(\mathbf{H}^3)$ and a discrete faithful representation ρ of Γ into $Isom(\mathbf{H}^n)$, for any $n \geq 4$, such that*

(1) *$\rho \in \partial CC^n(\Gamma)$ and*
(2) *$\rho(\Gamma)$ is a geometrically finite group that contains parabolics and*
(3) *the groups Γ and $\rho(\Gamma)$ are topologically conjugate on $\mathbf{H}^n$.*

Proof. We first prove (1) and (2). One can assume that $n = 4$. (Indeed, for

any $m < n$, the inclusion $\mathrm{Isom}(\mathbf{H}^m) \hookrightarrow \mathrm{Isom}(\mathbf{H}^n)$ takes geometrically finite groups to geometrically finite groups [Bo] and parabolics to parabolics).

We next present a spherical polyhedon in $\overline{\mathbb{R}^3}$ that is to be a fundamental polyhedron for $\rho(\Gamma)$.

Fix a plane L in $\mathbb{R}^3$ passing through the origin and consider a regular ν-gon in the plane centered at the origin where $\nu \geq 5$. To the vertices of the ν-gon we place metric balls of equal radii such that adjacent balls form an exterior angle $\pi/2$. By analogy, we call them *necklace balls*.

In addition, we consider two disjoint half spaces in $\mathbb{R}^3$ that are symmetric with respect to L and, moreover, each of them forms an exterior angle $\pi/3$ with any necklace ball. Note that these two half spaces are uniquely determined by L, in particular, their boundary planes are parallel to L. Any half space can be thought of as a conformal ball in $\overline{\mathbb{R}^3}$. We refer to the half spaces as the *big balls*; they are tangent at the point $\{\infty\} \in \overline{\mathbb{R}^3}$.

The complement in $\overline{\mathbb{R}^3}$ to the union of the necklace balls and big balls is a spherical polyhedron F. Consider a group generated by reflections in the sides of F. Since all dihedral angles of F are submultiples of π, the group is discrete and F is its fundamental polyhedron. Moreover the group is isomorphic to a cocompact reflection subgroup Γ of $\mathrm{Isom}(\mathbf{H}^3)$ described in 3.3 for $\alpha = \pi/2$ and $\beta = \pi/3$. This defines a faithful discrete representation ρ of Γ into $\mathrm{Isom}(\mathbf{H}^4)$. The group $\rho(\Gamma)$ is geometrically finite since F has finitely many sides. Also the composition of the reflections in the boundary spheres of big balls is a parabolic element. This proves (2).

To show (1), note that we can slightly perturb the big balls, so that after the perturbation, they are disjoint while all the angles are the same. The reflections in the sides of the perturbed spherical polyhedron generate a convex-cocompact group isomorphic to Γ. This defines a sequence $\rho_k \in CC^4(\Gamma)$ that converges to ρ.

Finally, let us prove (3). Given an inclusion $\mathbf{H}^m \subset \mathbf{H}^n$, any polyhedron $P \subset \mathbf{H}^m$ defines a unique polyhedron in $\mathbf{H}^n$ which is the pre–image of P under the orthogonal projection $\mathbf{H}^n \to \mathbf{H}^m$. Thus the groups Γ and $\rho(\Gamma)$ have preferred fundamental polyhedrons (chambers) in $\mathbf{H}^n$. It is easy to see that the chambers are face preserving homeomorphic. Therefore, by Theorem 3.4, Γ and $\rho(\Gamma)$ are topologically conjugate on $\mathbf{H}^n$. $\square$

4.11. Remark. One can easily generalize the construction above to produce more boundary groups. For instance, we can start with a knotted necklace whose balls are orthogonal to a hyperplane in $\mathbb{R}^4$; then we add two big balls that are disjoint half spaces in $\mathbb{R}^4$ symmetric with respect to the hyperplane. If necessary, we also add filling balls. If the angles are chosen properly, the group generated by reflections in the boundary spheres of the balls is discrete and isomorphic to a lattice in $\mathrm{Isom}(\mathbf{H}^3)$. Another idea is to define more sophisticated side pairing transformations, similarly to what was done in 4.2.

§5. Core submanifolds and the Euler class

Let Γ be a discrete convex-cocompact group satisfying the assumption of 4.8 and let G be a torsion-free subgroup of Γ of finite index (which exists by the Selberg lemma). Passing to a subgroup of index two, if necessary, we can assume G preserves orientation. Since Γ is isomorphic to a cocompact lattice in $\mathrm{Isom}(\mathbf{H}^3)$, the group G is the fundamental group of a closed oriented hyperbolic 3-manifold M (which is uniquely determined up to an isometry by the Mostow Rigidity Theorem).

In this section we construct an embedding $M \hookrightarrow_i N = \mathbf{H}^5/G$ inducing a homotopy equivalence. We next study the Euler class χ_M^N of the embedding (or, equivalently, the normal Euler class of $i(M)$ in N). In particular, we shall prove the following

5.1. Theorem. *There exists a closed oriented hyperbolic 3-manifold M and a topological embedding of M into some complete oriented hyperbolic 5-manifold N such that*

(i) *the Euler class χ_M^N of the embedding is nonzero;*
(ii) *the embedding induces a homotopy equivalence.*

5.2. Euler class and intersections

Let $M = M^n$ and $N = N^{n+k}$ be orientable manifolds without boundary. Let $M \hookrightarrow_i N$ be a proper (topological) embedding.

According to a standard definition [D], the Thom class τ_M^N of the embedding i is a generator of the group $\mathrm{H}^k(N, N \setminus M; \mathbb{Z}) \simeq \mathbb{Z}$ chosen in a natural way. The image of τ_M^N under the homomorphism $\mathrm{H}^k(N, N \setminus M; \mathbb{Z}) \to \mathrm{H}^k(M; \mathbb{Z})$ induced by the composition of the inclusions $M \hookrightarrow_i N \hookrightarrow_j (N, N \setminus M)$ is said to be the Euler class χ_M^N of the embedding.

There is a version of the Poincaré duality for noncompact manifolds (see e.g. [Mass]). In particular, there is a natural isomorphism of the exact sequences of the pairs $(N, N \setminus M)$ and (N, M):

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & H^k(N, N \setminus M; \mathbb{Z}) & \xrightarrow{j^*} & H^k(N; \mathbb{Z}) & \longrightarrow & H^k(N \setminus M; \mathbb{Z}) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle D'} & & \downarrow{\scriptstyle D} & & \downarrow & & \\
\ldots & \longrightarrow & H_n^\infty(M; \mathbb{Z}) & \xrightarrow[i_*]{} & H_n^\infty(N; \mathbb{Z}) & \longrightarrow & H_n^\infty(N, M; \mathbb{Z}) & \longrightarrow & \ldots
\end{array}$$

where H^* means the singular cohomology and H_*^∞ means a homology of infinite chains [Mass]; the vertical arrows are isomorphisms; moreover, D is the Poincaré isomorphism.

Now we present some calculations connecting the Euler class and the intersection number. We recall that, for homology classes $x \in H_n^\infty(N; \mathbb{Z})$ and $y \in H_k(N; \mathbb{Z})$, their intersection number $\langle x, y \rangle$ is equal to $z[y]$, where $Dz = x$. The group $H_n^\infty(M; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and generated by the fundamental class $[M]$ of M. So $D'(\tau_M^N)$ is equal to $[M]$, up to sign.

Let $f \in \mathrm{H}_k(M;\mathbb{Z})$. Then

$$\chi_M^N[f] = i^*j^*\tau_M^N[f] = j^*\tau_M^N[i_*f] = \langle i_*D'\tau_M^N,\, i_*f\rangle = \pm\langle i_*[M],\, i_*f\rangle.$$

Thus, the value of the Euler class χ_M^N on f is equal (up to sign) to the intersection number of M and f in N.

5.3. Embedding of M into N.

We next construct a topological embedding of M into N.

Consider the hyperbolic polyhedron P in $\mathbf{H}^5$ which is a fundamental polyhedron for Γ. Let l be the geodesic in $\mathbf{H}^5$ joining 0 and ∞. There is a natural order on l such that $0 < \infty$ and $t_1 < t_2 < t_3$ if t_2 belongs to the interval (t_1, t_3). For $t \in l$, denote by P_t the hyperplane in $\mathbf{H}^5$ meeting l in the point t and orthogonal to l. Put $[a,d] = l \cap P$ where $a < d$. Each of the hyperplanes P_a and P_b contains a b-side of P. The construction of P implies that there are two points $b, c \in (a,d)$ with $b < c$ such that any vertex of P lies in one of the hyperplanes P_a, P_b, P_c or P_d. Notice also that any f-side of P either lies between the hyperplanes P_a and P_b or lies between P_c and P_d. In the former case we call such an f-side a *smaller* f-side of P.

In each vertex of P we choose a point in such a way that if two vertices are Γ-equivalent then the points chosen are Γ-equivalent too. For each pair of vertices lying on the same edge, consider the unique geodesic segment connecting the the points chosen in the vertices. We get a piecewise-geodesic graph γ.

First, we observe that $C_a = \gamma \cap P_a$ and $C_d = \gamma \cap P_d$ are piecewise-geodesic circles. Indeed, by the construction of P, C_a and C_d we get $C_d = i(C_a)$ where i is the extension to $\mathbf{H}^5$ of the inversion in the unit sphere in $\mathbb{R}^4$. So it suffices to show C_a is a circle. Consider the set of edges of P lying on P_a. Any such an edge meets exactly two other edges of the set and the intersection of any two edges of the set is either the empty set or a vertex of P. This implies that every edge of the graph C_a meets exactly two other edges of the graph. Hence C_a is a circle.

For $t \in (a,d)$, the intersection γ_t of γ and P_t is a finite set of points. For any two points of γ_t lying on the same side of P, we consider the geodesic segment connecting the points. Since any side is convex the segment is contained in the side. The union of all such segments is a piecewise-geodesic graph C_t. We now show that, for any $t \in (a,d)$, the graph C_t is a circle.

First, we assume that $t \in (a,b)$. Then the hyperplane P_t meets every necklace side and each smaller f-side. Notice that any edge of P meeting P_t is the (only) common edge of a necklace side and a smaller f-side. Moreover, once P_t meets a side of P, it meets exactly two edges of P bounding the side. Hence every side of P meeting P_t contains exactly two points of γ_t and every point of γ_t lies in exactly two sides. Thus C_t is a circle. Applying the inversion i we also get that C_t is a circle for $t \in (c,d)$.

Second, suppose $t = b$. The hyperplane P_b meets any necklace side. Also P_b intersects any smaller f-side so that the intersection is a vertex of P, the

only common vertex of this f-side and two adjacent necklace sides. Every such a vertex contains exactly one point of γ_b and any point of γ_b lies in one of such vertices. So any necklace side contains exactly two points of γ_b and any point of γ_b belongs to precisely two necklace sides. Hence C_b is a circle. We also get $C_c = i(C_b)$ is a circle.

Finally, assume that $t \in (b, c)$. Then P_t meets only (and each of) the necklace sides. Any point of γ_t lies in precisely two necklace sides and any necklace side contains exactly two points of the set γ_t. So C_t is a circle.

For $t \in [a, b]$, denote by D_t the geodesic cone from t over the C_t. Finally, let D be the union of all D_t, for $t \in [a, b]$.

5.4. Example. Let Γ_0 be a discrete group constructed in 3.3 or 4.5 that is isomorphic to Γ. The group Γ_0 stabilizes a 3-plane $\mathbf{H}^3$. We use the following notation convention: the index 0 is assigned to all objects concerning Γ_0. Thus P_0 is the fundamenal polyhedron for the action of Γ_0 in $\mathbf{H}^5$ and $D_0 \subset P_0$ is the 3-disc constructed in 5.3. It is easy to see that $D_0 = P_0 \cap \mathbf{H}^3$ provided the points chosen in the vertices of P_0 lie in $\mathbf{H}^3$. Clearly, D_0 is a fundamental polyhedron for the action of Γ_0 in $\mathbf{H}^3$.

Now we present a homeomorphism f from D_0 onto D equivariant with respect to the obvious isomorphism of Γ_0 and Γ. The construction of f is parallel to that of D.

First, we define f on the vertices of D_0. We consider the polyhedrons D_0 and P and fix a one-to-one correspondence of the sets of their sides, edges and vertices preserving the incidence. For a vertex v of D_0, let $f(v)$ be the unique point of D lying on the vertex of P corresponding to v.

We say that f proportionally maps a geodesic segment $[x, y]$ onto the geodesic segment $[f(x), f(y)]$ if the ratio of the distance between x and s to the distance between $f(x)$ and $f(s)$ does not depend on $s \in [x, y]$. Clearly, for any pair of geodesic segments $[x, y], [x', y']$, there is only one mapping f with $f(x) = x'$ carrying $[x, y]$ onto $[x', y']$ proportionally.

Now we define f on the edges of D_0. Let vertices x and y bound the edge e of D_0. Then $f(x)$ and $f(y)$ lie on the vertices of a certain edge of P. Let f proportionally map e to the geodesic segment connecting $f(x)$ and $f(y)$.

We next proceed in the same way. For every $t \in (a_0, d_0)$, we consider the piecewise-geodesic circle C^0_t. The mapping f has been already defined on the vertices of C^0_t. Let f proportionally map its edges onto the edges of C_t. In addition, let f proportionally map the segment $[a_0, d_0]$ onto $[a, d]$. Finally, for each $t \in [a_0, d_0]$, consider the geodesic cone D^0_t. We have defined f in its vertex t and on its base C^0_t. For any point x of C^0_t, let f proportionally map the geodesic segment $[x, t]$ onto the geodesic segment connecting $f(x)$ and $f(t)$. Obviously, the mapping f is a homeomorphism. That f is equivariant can be checked by direct inspection of side pairing transformations.

Extend f by equivariance to an (equivariant) homeomorphism $\tilde{f} : \mathbf{H}^3 \to \tilde{D}$, where $\tilde{D}$ is the union of all Γ-images of D. Moreover, by [Tu], $\tilde{f}$ can be extended to an equivariant homeomorphism $\hat{f} : \mathbf{H}^3 \cup \Lambda(\Gamma_0) \to \tilde{D} \cup \Lambda(\Gamma)$.

The group G acts freely in $\mathbf{H}^5$ and $\mathbf{H}^3$, so we get a proper embedding i of the closed hyperbolic 3-manifold $M = \mathbf{H}^3/G$ into the hyperbolic 5-manifold $N = \mathbf{H}^5/G$. Clearly, the embedding induces an isomorphism of fundamental groups, hence it induces a homotopy equivalence. Thus we have proved the part (ii) of Theorem 5.1.

5.5. Totally geodesic hypersurfaces

Now we consider a subgroup H of Γ generated by transformations pairing necklace sides. Such subgroups have been studied in [GLT] and [Kui1,2]. The group H is convex-cocompact and isomorphic to a uniform lattice in $\mathbf{PSL}_2(\mathbb{R})$, so the limit set $\Lambda(H)$ is a topological circle.

The group H acts by hyperboic isometries on the unit ball $\mathbb{B}^4$ in $\mathbb{R}^4$, hence $\Lambda(H) \subset \mathbf{S}^3$. Let $\mathbf{H}^4$ be the hyperplane in $\mathbf{H}^5$ spanning $\mathbf{S}^3$. Clearly, $P_H = P \cap \mathbf{H}^4$ is a fundamental polyherdron for the action of H in $\mathbf{H}^4$. Note that $\mathbf{H}^4$ lies in the interior of the union of all H-images of P. It follows that $\mathbf{H}^4$ is precisely invariant under the subgroup H in Γ (i.e. $H = \mathrm{Stab}_\Gamma \mathbf{H}^4$ and $g(\mathbf{H}^4) \cap \mathbf{H}^4 = \emptyset$ for all $g \in \Gamma \setminus H$). Put $K = H \cap G$. Then the hyperbolic 4-manifold $L = \mathbf{H}^4/K$ discovered in [GLT], [Kui1,2] is an incompressible totally geodesic hypersurface in $\mathbf{H}^5/G$.

Analogously, $F_H = F \cap \mathbf{S}^3$ is a spherical fundamental polyhedron for the action of H in $\mathbb{S}^3$. Then the union of all H-images of the set F_H is the discontinuity set Ω_H for the action of H in $\mathbf{S}^3$. We again note that Ω_H lies in the interior of the union of all H-images of F. Therefore Ω_H is precisely invariant under the subgroup H in G. Hence the closed conformally flat 3-manifold Ω_H/K discovered in [GLT], [Kui1,2] is a hypersurface in the closed conformally flat 4-manifold Ω/G.

5.6. The normal Euler class of M in N

We consider the 2-disk $D_H = D \cap \mathbf{H}^4$. It is easy to construct a uniform lattice in $\mathbf{PSL}_2(\mathbb{R})$ acting in $\mathbf{H}^2$ and isomorphic to H such that there is an equivariant homeomorphism of its fundamental polygon onto D_H. It defines an equivariant embedding of $\mathbf{H}^2$ into $\mathbf{H}^4$. Passing to quotients by $K = H \cap G$ we get a closed orientable surface $S = L \cap M$; the surface S is a deformation retract of the manifold L (see [GLT], [Kui1,2]).

It is easy to see that the intersection number $\langle M, S\rangle_N$ of M and S in N is equal to the self-intersection number $\langle S, S\rangle_L$ of S in L. In fact, the self-intersection number of S in L has been calculated in [GLT] and [Kui1,2]. It follows from [GLT] that if Γ comes from §3 then, for a corresponding surface S, we get $\langle S, S\rangle_L = 0$.

However, for some groups of 4.4 the self-intersection numbers can be nonzero (see [Kui1,2]). In fact, if the necklace comes from 4.6, then $\langle S, S\rangle_L = pq - T$ where p and q are relatively prime intergers with $1 \le q < p$ and T is an arbitary integer such that $q < T < p$. Thus we have proved the part (i) of Theorem 5.1.

§6. Hyperbolic 5-manifolds and conformally flat 4-manifolds

In this section we study topological properties of hyperbolic 5-manifolds and of conformally flat 4-manifolds constructed above. Let Γ be a discrete convex-cocompact group satisfying the assumption of 4.8 and let $G \cong \pi_1(M)$ be a torsion-free subgroup of Γ of finite index that preserves orientation.

Consider the embedded necklace used in the construction of Γ. The intersection of the unit sphere $\mathbf{S}^3$ and the union of all necklace balls is a solid torus in $\mathbf{S}^3$. The cases of the knotted and unknotted solid tori lead to different families of manifolds.

6.1. Theorem. *If the solid torus determined by the necklace is unknotted, then the Kleinian manifold $M(G) = (\mathbf{H}^5 \cup \Omega)/G$ is homeomorphic to the total space of a 2-disc bundle over M. In particular the hyperbolic manifold $N = \mathbf{H}^5/G = int(M(G))$ is a plane bundle over M and the conformally flat manifold $\Omega/G = \partial M(G)$ is the associated circle bundle over M.*

Proof. It suffices to construct a Γ-compatible 2-disc fibration of the fundamental polyhedron $P \cup F$ over the disk D. (Indeed, having the fibration constructed we obtain a Γ-compatible fibration of $\mathbf{H}^5 \cup \Omega$ over $\tilde{D}$. Hence $M(G) = (\mathbf{H}^5 \cup \Omega)/G$ is the total space of a 2-disc bundle over $M = \tilde{D}/G$.)

Given a totally geodesic plane E in $\mathbf{H}^n$ we denote $\overline{E}$ the closure of E in $\mathbf{H}^5 \cup \partial_\infty \mathbf{H}^5 = \overline{\mathbf{H}^5}$. We say $\overline{E}$ is a *closed plane*.

We construct the fibration in question as follows. Let $t \in [a, d]$. Consider an arbitrary edge E of the polyhedron $P \cup F$ that does not lie on $\overline{P_a} \cup \overline{P_d}$. Then the intersection $E_t = E \cap \overline{P_t}$ is a closed 2-plane. (Notice, in particular, that each of the two vertices of $P \cup F$ that bounds E is of the form E_t for some $t \in \{a, b, c, d\}$.) Also, $E_t \cap D$ is a point. We define the 2-disc E_t to be a fiber over the point $E_t \cap D$. Now we take an arbitrary side S of $P \cup F$ which is not a b-side. Then $S \cap \overline{P_t}$ is a closed 3-plane with two disjoint half spaces removed; these half spaces are bounded by closed 2-planes of the form E_t and E'_t where E and E' are edges of P lying on ∂S. In the closed 3-plane we draw a unique geodesic line orthogonal to both E_t and E'_t and consider the family of closed 2-planes orthogonal to the line. The intersection of D and the closed 3-plane is a geodesic segment with end points $D \cap E_t$ and $D \cap E'_t$. For a closed 2-plane of the family, we take the common point of the plane and the segment and consider this closed plane to be a fiber over the point. Thus, we have constructed a 2-disc fibration $U \to U \cap D$ where U is the union of all the sides of $P \cup F$ except for two b-sides. It is clear that the fibration is Γ-compatible.

Consider the closed hyperplane $\overline{\mathbf{H}^4}$ in $\overline{\mathbf{H}^5}$ spanning the unit sphere $\mathbf{S}^3 \subset \mathbb{R}^4$. The hyperplane cut $P \cup F$ into two pieces P_+ and P_-. Note that the inversion i in the hyperplane $\overline{\mathbf{H}^4}$ takes P_+ to P_- preserving the fibration structure.

Since the solid torus determined by the necklace is unknotted, one can extend the fibration $U \to U \cap D$ to a 2-disc fibration $P_+ \cup U \to P_+ \cup U \cap D$.

Finally, define a fibration structure on P_- via the inversion i. Thus we have constructed a Γ-compatible 2-disc fibration of $P \cup F$ over D. $\square$

6.2. Hyperbolic manifolds with nonlocally-flat cores.

Assume that the solid torus determined by the necklace is knotted. Then the embedding $D \hookrightarrow P$ is not locally flat at the points of the segment $[a, b]$. (Indeed, let B_s be a small enough metric ball centered at a point $s \in (a, b)$ and let x, y be the common points of ∂B_s and (a, b). Then $\partial B_s \cap D$ is a 2-knot in the 4-sphere ∂B_s; the knot is the suspension over a 1-knot $\partial B_s \cap D_s$ with vertices x and y. The knot $\partial B_s \cap D_s$ belongs to a well defined isotopy knot class specified by the original solid torus. Sinilarly, at the points a and b the embedding is not locally flat relative to the boundary.) Therefore, if the solid torus is knotted, the embedding $M \hookrightarrow N = \mathbf{H}^5/G$ is not locally flat at the points of k disjoint circles, where $k = |\Gamma : G|$.

Recall that U is the union of all the sides of $P \cup F$ except for two b-sides. In 6.1 we constructed a 2-disc fibration of U. Denote by U_ε the ε-neighborhood of U in $P \cup F$ with respect to any Riemannian metric on $P \cup F$. For ε small enough, we can extend the fibration defined on U to a Γ-compatible fibration of U_ε.

Consider the union of all Γ-images of $(P \cup F) \setminus U_\varepsilon$. The quotient space of the union by the action of G is the disjoint union of k manifolds homeomorphic to $(P_H \cup F_H) \times S^1$, where $P_H = P \cap \mathbf{H}^4$ and $F_H = F \cap \mathbf{S}^3$ (the notation is taken from 5.6). The quotient mapping takes the union of all Γ-images of $D \setminus U_\varepsilon$ to a collection of k solid tori of the form $D_H \times S^1$ where $D_H = D \cap P_H$.

Now we consider the union of all Γ-images of U_ε. Its quotient space by the action of G is the total space N_0 of a plane bundle over M_0 where M_0 is M with k solid tori removed. Thus the manifold N is obtained by the pasting of k manifolds of the form $P_H \times S^1$ in N_0 along their boundaries. Each of the manifolds of the form $P_H \times S^1$ is glued to the corresponding boundary component of N_0 in such a way that the torus $\partial D_H \times S^1$ is glued to the boundary torus of M_0.

Let E be the closed conformally flat 4-manifold Ω/G, where Ω is the discontinuity set for G. It follows from the considerations above that E can be obtained by gluing along the boundaries k manifolds of the form $F_H \times S^1$ and the total space E_0 of a circle bundle over M_0. (Note that F_H is the comlpement in $\mathbf{S}^3$ of the solid torus determined by the original necklace.)

§7. Embeddings of limit sets

Throughout this section, let Γ be a discrete convex-cocompact group satisfying the assumption of 4.8 and let G be a torsion-free subgroup of Γ of finite index that preserves orientation.

7.1. Theorem. *If the solid torus determined by the necklace is unknotted then the limit set $\Lambda(\Gamma)$ is an unknotted 2-sphere in $\mathbb{S}^4$.*

Proof. We need to show that $\Lambda = \Lambda(\Gamma) = \Lambda(G)$ is unknotted. The manifold

$E = \Omega/G$ is the total space of a circle bundle ξ over a closed hyperbolic 3-manifold M with $\pi_1(M) \simeq G$. The circle bundle ξ is orientable since the manifolds E and M are orientable.

First, consider the exact sequence of the bundle:

$$1 \to \pi_1(S^1) \to \pi_1(E) \to \pi_1(M) \to 1.$$

Let t be an element of $\pi_1(E)$ corresponding to a fiber of the bundle (all the fibers are freely homotopic). Since ξ is orientable t lies in the center of $\pi_1(E)$. Moreover, t generates the center of $\pi_1(E)$ because the center of $\pi_1(M)$ is trivial.

Now consider the exact sequence of the regular covering $\Omega \to_p E$:

$$1 \to \pi_1(\Omega) \xrightarrow{p_*} \pi_1(E) \to G \to 1.$$

Since the center of G is trivial, the subgroup $p_*(\pi_1(\Omega))$ contains the center of $\pi_1(E)$. Moreover, $p_*(\pi_1(\Omega))$ has to coincide with the center of $\pi_1(E)$ because G is Hopfian (any finitely generated subgroup of $\mathbf{GL}(n, \mathbb{C})$ is residually finite, hence Hopfian). Consider the regular covering $\mathbb{B}^3 \times S^1 \to E$ induced by the universal covering $\mathbb{B}^3 \to M$. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{B}^3 \times S^1 & \longrightarrow & \mathbb{B}^3 \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} .$$

The regular covering $\mathbb{B}^3 \times S^1 \to E$ is isomorphic to the regular covering $\Omega \to_p E$ since they correspond to the same subgroup of $\pi_1(E)$, namely its center. In other words, there is a G-compatible homeomorphism $\Omega \simeq \mathbb{B}^3 \times S^1$. Therefore the natural $\mathbf{S}^1$-action on E whose orbits are the fibers of E can be lifted to a G-compatible $\mathbf{S}^1$-action on Ω. By the Tukia Theorem [Tu], the $\mathbf{S}^1$-action can be extended to the trivial action on $\Lambda(G)$.

We next note that the projection from $\mathbf{S}^4$ onto the orbit space of the $\mathbf{S}^1$-action coincides with the mapping $\pi : \mathbf{S}^4 \to \mathbb{B}^3 \cup \partial\mathbb{B}^3$ defined as follows. Let $\pi \mid_\Omega$ be the composition of the G-compatible mappings $\Omega \simeq \mathbb{B}^3 \times \mathbf{S}^1 \to \mathbb{B}^3$, where the latter mapping is taken from the commutative diagram above. In addition, let $\pi \mid_\Lambda$ be the unique G-compatible homeomorphism of Λ onto the infinity $\partial\mathbb{B}^3$ of the hyperbolic 3-space $\mathbb{B}^3$ constructed in [Tu]. By a lemma of Tukia [Tu], π is continuous. It easily implies that π coincides with the projection.

Finally, we show that the $\mathbf{S}^1$-action is equivalent to an orthogonal one. It suffices to construct a section s of the projection π (see [Br, I.3.4]). The restriction of s on $\mathbb{B}^3$ is defined to be a section of the trivial circle fibration $\Omega \to \mathbb{B}^3$. (Note that the section is not required to be G-compatible.) The restriction of s on $\partial\mathbb{B}^3$ is defined to be $(\pi \mid_\Lambda)^{-1}$. Clearly, the set $\mathrm{Im}(s)$

is closed in $\mathbf{S}^4$ and contains exactly one point from each orbit. Hence s is a section (see [Br, I.3.2]). Thus the $\mathbf{S}^1$-action considered is equivalent to orthogonal one and there is a homeomorphism of $\mathbf{S}^4$ carrying Λ to a round 2-sphere. □

7.2. Remark. The first part of the proof of 7.1 concerning the lifting of the $\mathbf{S}^1$-action to $\mathbf{S}^4$ analogous to Kapovich's arguments for the 3-dimensional case [Ka1]. However, to prove that the $\mathbf{S}^1$-action is equivalent to an orthogonal one, Kapovich made use of Raymond's classification of $\mathbf{S}^1$-actions on $\mathbf{S}^3$. Our arguments are more elementary and do not depend on the dimension.

7.3. Uniformly quasiconformal groups

Applying Kapovich's method [Ka1] we now construct a family of discrete uniformly quasiconformal groups which are not topologically conjugated to conformal ones (cf. [FS], [Mart]). By the Tukia Theorem [Tu], the circle action on $\mathbf{S}^4$ constructed in Theorem 5.1 is uniformly quasiconformal. We regard $\mathbb{Z}_n$ as a subgroup of $\mathbf{S}^1$ generated by a rotation of order n. The G-action on $\mathbf{S}^4$ is conformal, hence uniformly quasiconformal. It defines an action of $G \times \mathbb{Z}_n$ on $\mathbf{S}^4$ that is also uniformly quasiconformal. If the action is equivalent to a conformal one then one can easily conclude $E = M \times \mathbf{S}^1$. Since some of the bundles E are nontrivial we have obtained discrete uniformly quasiconformal groups which are not topologically conjugated to conformal ones.

7.4. Corrolary. *There is a closed orientable hyperbolic 3-manifold M and an (effective) uniformly quasiconformal action of the group $\pi_1(M) \times \mathbb{Z}_n$ on $\mathbf{S}^4$ that is not topologically conjugate to a conformal action. The action is properly discontinuous on the complement to an unknotted 2-sphere which is the limit set of the action.*

7.5. Theorem. *If the solid torus determined by the necklace is knotted then the limit set $\Lambda(\Gamma)$ is a wild 2-sphere in $\mathbf{S}^4$.*

Proof. It follows from 6.2 that there exists a finite family of k disjoint 3-tori in E such that, cutting E along the tori, we get E_0 and k manifolds of the form $F_H \times S^1$. We next note that the 3-tori are incompressible, that is, the inclusion of each of the tori into E induces a monomorphism of the fundamental groups.

By Dehn's lemma, ∂F_H is incompressible in F_H; hence $\partial F_H \times S^1$ is incompressible in $F_H \times S^1$. It remains to show that the boundary of E_0 is incompressible. Clearly, it suffices to check that the base M_0 of the circle bundle E_0 has the incompressible boundary.

Assume the boundary of M_0 is compressible. Take a nontrivial loop γ on a boundary torus T of M_0 such that γ is null-homotopic in M_0. Consider the coverings $\tilde{T} \subset \tilde{M}_0$ induced by the universal covering $\tilde{D} \to M$. Then γ can be lifted to a loop $\tilde{\gamma}$ in $\tilde{T}$. It is clear from the way we constructed M_0 that $\tilde{T}$ (as well as all other boundary component of $\tilde{M}_0$) is a cylinder. Notice that the

loop $\tilde{\gamma}$ must be nontrivial on $\tilde{T}$ and null-homotopic in $\tilde{M}_0 \subset \tilde{D} \setminus \mathrm{int}(\tilde{T})$. On the other hand, the cylinder $\tilde{T}$ is incompressible in the solid torus $\tilde{D} \setminus \mathrm{int}(\tilde{T})$. This gives a contradiction desired.

Thus the 3-tori considered are incompressible in E; hence, by the Van Kampen theorem, the composition of inclusions $F_H \hookrightarrow F_H \times S^1 \hookrightarrow E$ induces a monomorphism of the fundamental groups.

Since the inclusion $M \hookrightarrow N \cup E$ is a homotopy equivalence, there exists a deformation retraction $r\colon N \cup E \to M$. Moreover, this retraction can be chosen so that it maps $P_H \cup F_H$ onto the 2-disk $D_H \subset M$ (here we invoke the decomposition of $N \cup E$ described in 6.2). It gives rise to a G-compatible retraction $\tilde{r}\colon \mathbf{H}^5 \cup \Omega \to \tilde{D}$. Restricting $\tilde{r}$ to Ω we get the commutative diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\pi} & E \\ {\scriptstyle \tilde{r}}\downarrow & & \downarrow{\scriptstyle r} \\ \tilde{D} & \xrightarrow[p]{} & M \end{array} ,$$

where the vertical arrows are the retractions and the horizontal ones are the coverings. We now show that

$$\mathrm{Ker}(r_*\colon \pi_1(E) \to \pi_1(M)) = \mathrm{Im}(\pi_*\colon \pi_1(\Omega) \to \pi_1(E)).$$

Since $\pi_1(\tilde{D}) = 1$ the map $r \circ \pi = p \circ \tilde{r}$ induces the trivial homomorphism of the fundamental groups. Thus $\mathrm{Im}(\pi_*) \subset \mathrm{Ker}(r_*)$. To prove the opposite inclusion we take a loop α in E such that $[\alpha] \in \pi_1(E) \setminus \mathrm{Im}(\pi_*)$. Then the lift $\tilde{\alpha}$ of α to the covering space Ω is a path whose ends are G-equivalent. Since $\tilde{r}$ is G-compatible, the path $\tilde{r}(\tilde{\alpha})$ also has G-equivalent ends. So $p \circ \tilde{r}(\tilde{\alpha})$ is a nontrivial loop in $\pi_1(M)$. Hence $r(\alpha) = r \circ \pi(\tilde{\alpha}) = p \circ \tilde{r}(\tilde{\alpha})$ is a nontrivial loop in $\pi_1(M)$. Thus $[\alpha] \notin \mathrm{Ker}(r_*)$ and, therefore, $\mathrm{Ker}(r_*) \subset \mathrm{Im}(\pi_*)$.

Since the retraction r maps F_H onto the 2-disk D_H the subgroup $\pi_1(F_H)$ of the group $\pi_1(E)$ lies in $\mathrm{Ker}(r_*\colon \pi_1(E) \to \pi_1(M))$. As F_H is a nontrivial knot space the group $\pi_1(F_H)$ is nonabelian; hence the group

$$\mathrm{Ker}(r_*\colon \pi_1(E) \to \pi_1(M)) = \mathrm{Im}(\pi_*\colon \pi_1(\Omega) \to \pi_1(E)) \simeq \pi_1(\Omega)$$

is nonabelian.

Therefore the sphere $\Lambda = \Lambda(\Gamma)$ is knotted. We next show that Λ cannot be tame. Assume Λ is tame. Then Λ must be locally flat. (Indeed, if Λ were not locally flat at a point $x \in \Lambda$ then Λ would be not locally flat at each point of the infinite Γ-orbit of x; it would follow the wildness of Λ.) Then Ω is the interior of a compact manifold whose boundary is homeomorphic to $S^2 \times S^1$. Hence Ω is dominated by $S^2 \times S^1$ [Kul]. In particular, $\pi_1(\Omega) \leq \pi_1(S^2 \times S^1) \simeq \mathbb{Z}$ and the group $\pi_1(\Omega)$ is Abelian. A contradiction. Thus Λ is a wild knot and we are done. $\square$

§8. A theorem on convex-cocompact representations

In this section we prove Theorem 1.4. It follows from either Theorem 1.1 or Theorem 1.3 assuming the following folklore result on convex-cocompact groups.

Let Γ be a torsion free convex-cocompact subgroup of $G = \mathrm{Isom}_+(\mathbf{H}^n) = \mathbf{SO}_0(n,1)$; so Γ is finitely presented. We equip $\mathrm{Hom}(\Gamma, G)$ with pointwise convergence topology.

8.1. Theorem. *The subset $CC^n(\Gamma) \subset Hom(\Gamma, G)$ is open. Moreover, given representations ρ_1 and ρ_2 that lie in the same connected component of $CC^n(\Gamma)$, there is a self-homeomorphism f of $\mathbf{H}^n \cup \partial_\infty \mathbf{H}^n$ such that $f\rho_1(\gamma)f^{-1} = \rho_2(\gamma)$ for all $\gamma \in \Gamma$.*

Proof. It suffices to show that for every $\rho \in CC^n(\Gamma)$, there exsists an open neighborhood U_ρ of ρ in $\mathrm{Hom}(\Gamma, G)$ such that all $\phi \in U_\rho$ are convex-cocompact and topologically conjugate to the representation ρ on $\mathbf{H}^n \cup \partial_\infty \mathbf{H}^n$. (Indeed, it implies that $CC^n(\Gamma)$ is open and that any two representations that lie in the same path-connected component of $CC^n(\Gamma)$ are topologically conjugate. It remains to observe that every connected component of $CC^n(\Gamma)$ must be path-connected. Since $\mathrm{Hom}(\Gamma, G)$ is a real-algebraic variety, it is homeomorphic to a simplicial complex. Being an open subset of $\mathrm{Hom}(\Gamma, G)$, the space $CC^n(\Gamma)$ is locally path-connected. So any connected component of $CC^n(\Gamma)$ is path-connected.)

Start with a representation $\rho \in CC^n(\Gamma)$ and denote $R = \rho(\Gamma)$. First, assume that $\mathbf{H}^n/R$ is a closed manifold. According to [Go] one can find an open neighborhood U_ρ of ρ in $\mathrm{Hom}(\Gamma, G)$ such that each $\phi \in U_\rho$ is a holonomy of some hyperbolic structure on M. Since M is closed, the structure is complete [Th], in other words, it arises as $\mathbf{H}^n/F$ where $F = \phi(\Gamma)$. So F is convex-cocompact. The identity map of M lifts to a topological conjugacy of ϕ and ρ on $\mathbf{H}^n$. As both groups are convex-cocompact, it extends to a topological conjugacy of ϕ and ρ on $\mathbf{H}^n \cup \partial_\infty \mathbf{H}^n$ [Tu].

It remains to consider the case when R acts discontinuously at some point of $\partial_\infty \mathbf{H}^n$. According to [Sul], there exsists an open neighborhood U_ρ of ρ in $\mathrm{Hom}(\Gamma, G)$ such that, for all $\phi \in U_\rho$, there exists an equivariant homeomorphism of the limit set $\Lambda(R)$ onto a nearby compact $F = \phi(\Gamma)$-invariant subset of $\partial_\infty \mathbf{H}^n$. Clearly this compact subset contains $\Lambda(F)$ which is the smallest compact F-invariant set. (Note that we have not yet shown that F is discrete. Still the limit set of F is defined as the set of accumulation points of the F-orbit of any point of $\mathbf{H}^n$.) Conversely, any point of the compact subset is a limit point because of the conjugacy of the actions. Thus we have constructed a $\phi \circ \rho^{-1}$-equivariant homeomorphism of the limit sets. In particular, it follows that $\phi \circ \rho^{-1}$ and hence ϕ is injective.

Since the limit set $\Lambda(F)$ of F lies in a small neighborhood of $\Lambda(R) \neq \partial_\infty \mathbf{H}^n$, we conclude that $\Lambda(F)$ is a proper subset of $\partial_\infty \mathbf{H}^n$. In other words. the group F acts properly discontinuously at a point of $\partial_\infty \mathbf{H}^n$, in particular,

it is a discrete group. Being discrete, F has to be convex-cocompact because convex-cocompactness is encoded in the dynamics of the F-action on $\Lambda(F)$ which is equivalent to the action of the convex-cocompact group R on its limit set [Bo].

Look at the Kleinian manifold $M(R) = (\mathbf{H}^n \cup \Omega(R))/R$ where $\Omega(R) = \partial_\infty \mathbf{H}^n \setminus \Lambda(R)$ is the discontinuity set of R. This is a compact manifold with the boundary $\Omega(R)/R$. It has a natural conformal structure modeled on the space $(\overline{\mathbf{H}^n}, \partial_\infty \mathbf{H}^n) = (\overline{\mathbf{H}^n}, \mathbf{S}^{n-1})$ equipped with the standard $SO_0(n,1)$-action. We refer to the structure as to the Kleinian structure. According to [Go], one can make the open neighborhood U_ρ chosen above a little smaller so that each $\phi \in U_\rho$ is a holonomy of some Kleinian structure on $M(R)$. The developing map of such a structure is an equivariant map from the universal cover $(\tilde{M}(R), \partial\tilde{M}(R))$ to $(\overline{\mathbf{H}^n}, \mathbf{S}^{n-1})$ which is a local diffeomorphism.

In fact, one can choose U_ρ even smaller so that every $\phi \in U_\rho$ is a holonomy of a Kleinian structure on $M(R)$ such that the corresponding developing map takes $\partial\tilde{M}(R)$ to $\Omega(F)$. Indeed, since $\partial M(R)$ is compact, there is a compact subset K of $\partial\tilde{M}(R) = \Omega(R)$ that is mapped onto $\partial M(R)$ by the covering projection. Fix a nearby Kleinian structure on $(\tilde{M}(R), \partial\tilde{M}(R))$ with the holonomy group $F = \phi(\Gamma)$. Its developing map $\tilde{d}$ restricted to K is close to the inclusion $K \hookrightarrow \Omega(R)$. Recall that the limit set $\Lambda(F)$ lies in a small neighborhood of $\Lambda(R)$. Since the compact sets K and $\Lambda(R)$ are disjoint, so are $\Lambda(F)$ and $\tilde{d}(K)$. Therefore, $\Lambda(F)$ and $\tilde{d}(\partial\tilde{M}(R))$ are disjoint because $\Lambda(F)$ is F-invariant.

Passing to quotients we get a local diffeomorphism d of $M(R)$ to $(\mathbf{H}^n \cup \Omega(F))/F$. Since $M(R)$ is compact, so is $d(M(R))$. Moreover $d(M(R))$ is open, as d is a local diffeomorphism. Therefore, d is surjective. Double the compact manifolds $M(R)$ and $(\mathbf{H}^n \cup \Omega(H))/H$ along their boundaries and extend d to the doubles by symmetry. Then the extension $\overline{d}$ is a surjective local diffeomorphism of closed manifolds. So it is a covering map. The map $\tilde{d}$ is equivariant with respect to an isomorphism of $\rho(\Gamma)$ and H. It follows that $\overline{d}$ induces an isomorphism of fundamental groups, hence $\overline{d}$ is a diffeomorphism as desired. Putting together $\tilde{d}$ and the conjugacy on the limit sets we get [Tu] a topological conjugacy of $\rho(\Gamma)$ and H on $\mathbf{H}^n \cup \partial_\infty \mathbf{H}^n$. □

8.2. Remark. The same proof works for all negatively curved rank one symmetric spaces.

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