the augmented matrix. Now the (3, 3) entry in \( A \) is the only nonzero entry of the third row in the first \( n \) columns, so it can be scaled to 1 and then used as a pivot to zero out entries below it. Continuing in this way, \( A \) is eventually reduced to \( I \), by scaling each row with a pivot and then using only row operations that add multiples of the pivot row to rows below.

b. The row operations just described only add rows to rows below, so the \( I \) on the right in \( [A \ I] \) changes into a lower triangular matrix. By Theorem 7 in Section 2.2, that matrix is \( A^{-1} \).

21. Suppose \( A = BC \), with \( B \) invertible. Then there exist elementary matrices \( E_1, \ldots, E_p \) corresponding to row operations that reduce \( B \) to \( I \), in the sense that \( E_p \cdots E_2E_1 = I \). Applying the same sequence of row operations to \( A \) amounts to left-multiplying \( A \) by the product \( E_p \cdots E_1 \). By associativity of matrix multiplication,

\[
E_p \cdots E_1A = E_p \cdots E_1BC = IC = C
\]

so the same sequence of row operations reduces \( A \) to \( C \).

25. \( A = UDV^T \). Since \( U \) and \( V^T \) are square, the equations \( U^T U = I \) and \( V^T V = I \) imply that \( U \) and \( V^T \) are invertible, by the IMT, and hence \( U^{-1} = U^T \) and \( (V^T)^{-1} = V \). Since the diagonal entries \( \sigma_1, \ldots, \sigma_n \) in \( D \) are nonzero, \( D \) is invertible, with the inverse of \( D \) being the diagonal matrix with \( \sigma_1^{-1}, \ldots, \sigma_n^{-1} \) on the diagonal. Thus \( A \) is a product of invertible matrices. By Theorem 6, \( A \) is invertible and \( A^{-1} = (UDV^T)^{-1} = (V^T)^{-1}D^{-1}U^T = VD^{-1}U^T \).

**Answer to Checkpoint:** If \( A \) is an invertible \( n \times n \) matrix, with an LU factorization \( A = LU \), and if \( B \) is \( n \times p \), then \( A^{-1}B \) can be computed by first row reducing \( [L \ B] \) to a matrix \( [U \ Y] \) for some \( Y \) and then reducing \( [U \ Y] \) to \( [I \ A^{-1}B] \). One way to see that this algorithm works is to view \( A^{-1}B \) as \( [A^{-1}b_1 \cdots A^{-1}b_p] \) and use the LU algorithm to solve simultaneously the set of equations \( Ax = b_1, \ldots, Ax = b_p \). MATLAB uses this approach to compute \( A^{-1}B \) (after first finding \( L \) and \( U \)).

**Appendix: Permutated LU Factorizations**

Any \( m \times n \) matrix \( A \) admits a factorization \( A = LU \), with \( U \) in echelon form and \( L \) a permuted unit lower triangular matrix. That is, \( L \) is a matrix such that a permutation (rearrangement) of its rows (using row interchanges) will produce a lower triangular matrix with 1's on the diagonal.

The construction of \( L \) and \( U \), illustrated below, depends on first using row replacements to reduce \( A \) to a *permuted echelon form* \( V \) and then using row interchanges to reduce \( V \) to an echelon form \( U \). By watching the reduction of \( A \) to \( V \), we can easily construct a permuted unit lower triangular matrix \( L \) with the property that the sequence of operations changing \( A \) into \( U \) also changes \( L \) into \( I \). This property will guarantee that \( A = LU \). (See the paragraph before Example 2 in the text.)
The following algorithm reduces any matrix to a permuted echelon form. In the algorithm when a row is covered, we ignore it in later calculations.

1. Begin with the leftmost nonzero column. Choose any nonzero entry as the pivot. Designate the corresponding row as a pivot row.
2. Use row replacements to create zeros above and below the pivot (in all uncovered rows). Then cover that pivot row.
3. Repeat steps 1 and 2 on the uncovered submatrix, if any, until all nonzero entries are covered.

This algorithm forces each pivot to be to the right of the preceding pivots; when the rows are rearranged with the pivots in stair-step fashion, all entries below each pivot will be zero. Thus, the algorithm produces a permuted echelon matrix. Whenever a pivot is selected, the column containing the pivot will be used to construct a column of $L$, as we shall see.

As an example, choose any entry in the first column of the following matrix as the first pivot, and use the pivot to create zeros in the rest of column 1. We choose the (3, 1)-entry.

$$A = \begin{bmatrix}
1 & -1 & 5 & -8 & -7 \\
-2 & -1 & 4 & 9 & 1 \\
4 & 8 & -4 & 0 & -8 \\
2 & 3 & 0 & -5 & 3
\end{bmatrix} \sim \begin{bmatrix}
0 & -3 & 6 & -8 & -5 \\
0 & 3 & -6 & 9 & -3 \\
4 & 8 & 4 & 0 & -8 \\
0 & -1 & 2 & -5 & 7
\end{bmatrix} \leftarrow \text{1st pivot row}
$$

Row 3 is the first pivot row. Choose the (2, 2)-entry as the second pivot, and create zeros in the rest of column 2, excluding the first pivot row.

$$= \begin{bmatrix}
0 & -3 & 6 & -8 & -5 \\
0 & -3 & -6 & 9 & -3 \\
4 & 8 & 4 & 0 & -8 \\
0 & -1 & 2 & -5 & 7
\end{bmatrix} \sim \begin{bmatrix}
0 & 0 & 0 & 1 & -8 \\
0 & -3 & -6 & 9 & 3 \\
4 & 8 & 4 & 0 & -8 \\
0 & 0 & 0 & -2 & 6
\end{bmatrix} \leftarrow \text{2nd pivot row}
$$

$$\leftarrow \text{1st pivot row}$$

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Cover row 2 and choose the (4, 4)-entry as the pivot. (The row index of the pivot is relative to the original matrix.) Create zeros in the other rows (in the pivot column), excluding the first two pivot rows.

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & -8 \\
0 & 3 & 6 & 2 & 1 \\
4 & 8 & 3 & 0 & -8 \\
0 & 0 & 0 & 2 & 6
\end{bmatrix}
\begin{bmatrix}
\text{call this column c} \\
\text{column d}
\end{bmatrix}
\]

\[4 \text{th pivot row} \quad 2 \text{nd pivot row} \quad 1 \text{st pivot row} \quad 3 \text{rd pivot row}
\]

Let \( V \) denote this permuted echelon form, and permute the rows of \( V \) to create an echelon form. The first pivot row goes to the top, the second pivot row goes next, and so on. The resulting echelon matrix \( U \) is

\[
\begin{bmatrix}
4 & 8 & -4 & 0 & -8 \\
0 & 3 & -6 & 9 & -3 \\
0 & 0 & 0 & -2 & 6 \\
0 & 0 & 0 & 0 & 5
\end{bmatrix} = U
\]

The last step is to create \( L \). Go back and watch the reduction of \( A \) to \( V \). As each pivot is selected, take the pivot column, and divide the pivot into each entry in the column that is not yet in a pivot row. Place the resulting column into \( L \). At the end, fill the holes in \( L \) with zeros.

Column: a b c d

\[
\begin{bmatrix}
1 & -3 & 1 & 3 \\
-2 & 3 & 1 & -3 \\
4 & -1 & -2 & -5 \\
2 & -1 & -1/2 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1/4 & -1 & -1/2 & 1 \\
-1/2 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1/2 & -1/3 & 1 & 0
\end{bmatrix}
\]

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You can check that $LU = A$. To see why this is so, observe that $L$ is constructed so the operations that reduce $A$ to $V$ also reduce $L$ to a permuted identity matrix. Since the pivots in $L$ are in exactly the same rows as in $V$, the sequence of row interchanges that reduces $V$ to $U$ also reduces the permuted identity matrix to $I$. Thus, the full sequence of operations that reduces $A$ to $U$ also reduces $L$ to $I$, so that $A = LU$. (See argument at the bottom of page 125 of the text.)

The next example illustrates what to do when $V$ has one or more rows of zeros. The matrix is from the Practice Problem for Section 2.5. For the reduction of $A$ to $V$, pivots were chosen to have the largest possible magnitude (the choice used for “partial pivoting”). Of course, other pivots could have been selected.

$$
A = \begin{bmatrix}
2 & -4 & -2 & 3 \\
6 & -9 & -5 & 8 \\
2 & -7 & -3 & 9 \\
4 & -2 & -2 & -1 \\
-6 & 3 & 3 & 4
\end{bmatrix}
$$

$$
\sim V = \begin{bmatrix}
0 & 0 & 0 & 0 \\
6 & -9 & -5 & 8 \\
0 & 0 & 0 & 0 \\
0 & -6 & -2 & 12 \\
0 & -6 & -2 & 12
\end{bmatrix}
$$

$$
\sim U = \begin{bmatrix}
0 & 0 & 0 & -5/3 \\
6 & -9 & -5 & 8 \\
0 & 0 & 0 & 5/3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

The first three columns of $L$ come from the three pivot columns above.

$$
\begin{bmatrix}
2 \\
6 \\
2 \\
4 \\
-6
\end{bmatrix}
\begin{bmatrix}
-1 \\
4 \\
4 \\
-5/3 \\
5/3
\end{bmatrix}
+ \begin{bmatrix}
6 \\
-6 \\
-6 \\
6 \\
21
\end{bmatrix}
$$

The first three columns of $L$ come from the three pivot columns above.
The matrix $L$ needs two more columns. Use columns 1 and 3 of the $5 \times 5$ identity matrix to place 1’s in the "nonpivot" rows 1 and 3. Fill in the remaining holes with zeros.

$$
\begin{bmatrix}
\frac{1}{3} & 1/6 & -1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1/3 & 2/3 & -1 & 0 & 1 \\
2/3 & -2/3 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
- L = 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
2/3 & -2/3 & 1 & 0 & 0 \\
1/3 & 1/6 & -1 & 1 & 0 \\
1/3 & 2/3 & -1 & 0 & 1 \\
\end{bmatrix}
$$

Row reduction of $L$ using only row replacements produces a permuted identity matrix. Moving the 1’s in the "pivot rows" 2, 5, and 4 into rows 1, 2, and 3 of the identity requires the same row swaps as reducing $V$ to $U$. If a further row interchange on the permuted identity is required, it will involve the bottom two rows, which came from the "nonpivot" rows 1 and 3. A corresponding interchange of the bottom two rows of $U$ has no effect on $U$ (and the product $LU$ is unaffected). As a result, $L$ is reduced to $I$ by the same operations that reduce $A$ to $V$ and then to $U$. Check that $A = LU$.

**MATLAB LU Factorization and the Backslash Operator \**

Row reduction of $A$ using the command `gauss` will produce the intermediate matrices needed for an LU factorization of $A$. You can try this on the matrix in Example 2, stored as Exercise 33 in the Laydata Toolbox. The matrices in (5) on page 145 in the text are produced by the commands

- `U=gauss(A,1)`
  - U has 0's below the first pivot
- `U=gauss(U,2)`
  - Now U has 0's below pivots 1 and 2
- `U=gauss(U,3)`
  - The echelon form

You can copy the information from the screen onto your paper, and divide by the pivot entries to produce $L$ as in the text. For most text exercises, the pivots are integers and so are displayed accurately.

To construct a permuted LU factorization, use `U=gauss(U,r,v)`, where $r$ is the row index of the pivot and $v$ is a row vector that lists the rows to be changed by replacement operations. For example, if $A$ has 5 rows and the first pivot is in row 4, use `U=gauss(A,4,[1 2 3 5])`. If the next pivot is in row 2, use `U=gauss(U,2,[1 3 5])`. To build the permuted matrix $L$, use full columns from $A$ or the partially reduced $U$, divided by the pivots. Then change entries to zero if they are in a row already selected as a "pivot row."

The MATLAB command `[L U]=lu(A)` produces a permuted LU factorization for any square matrix $A$, but it does not handle the general case.